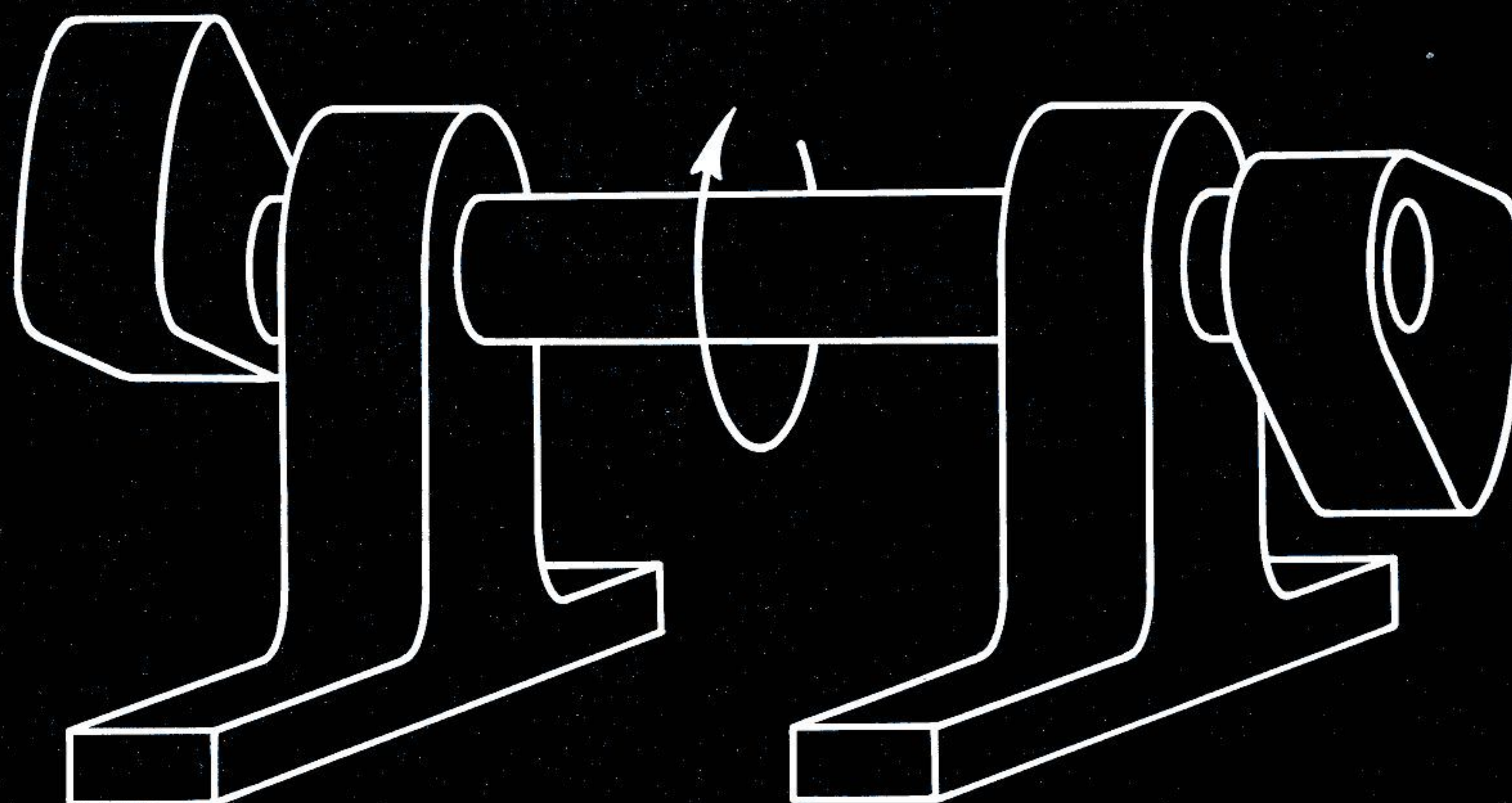


## Static and Dynamic Balancing



**Second Edition**



# Static and Dynamic Balancing using portable measuring equipment

by John Vaughan

## Foreword

Many people are needlessly apprehensive of performing their own dynamic balancing procedure. To help overcome these fears, this Application Note starts by showing how very simple and straightforward such a process can be when using B & K equipment.

## Installation

First attach two accelerometers, one near each of the bearings of the rotor being balanced, to measure vibration.

Mount a photoelectric trigger to give one pulse for each revolution of the rotor.

Connect the accelerometers via a changeover switch, to a vibration meter and hence to the "unknown" channel of a phase meter.

Connect the photoelectric trigger to the "known" channel of the phase meter.

## Establishment of original condition

Start the test rotor.

Note the amplitude shown on the vibration meter, and the angle on the phase meter for one of the planes (Plane 1).

Note the amplitude and angle shown for the other plane (Plane 2).

Stop the test rotor.

## Trial run 1

Fix a known test mass ( $M_1$ ) onto the rotor at the radius and in the plane where mass correction is to be made, nearest to Plane 1.

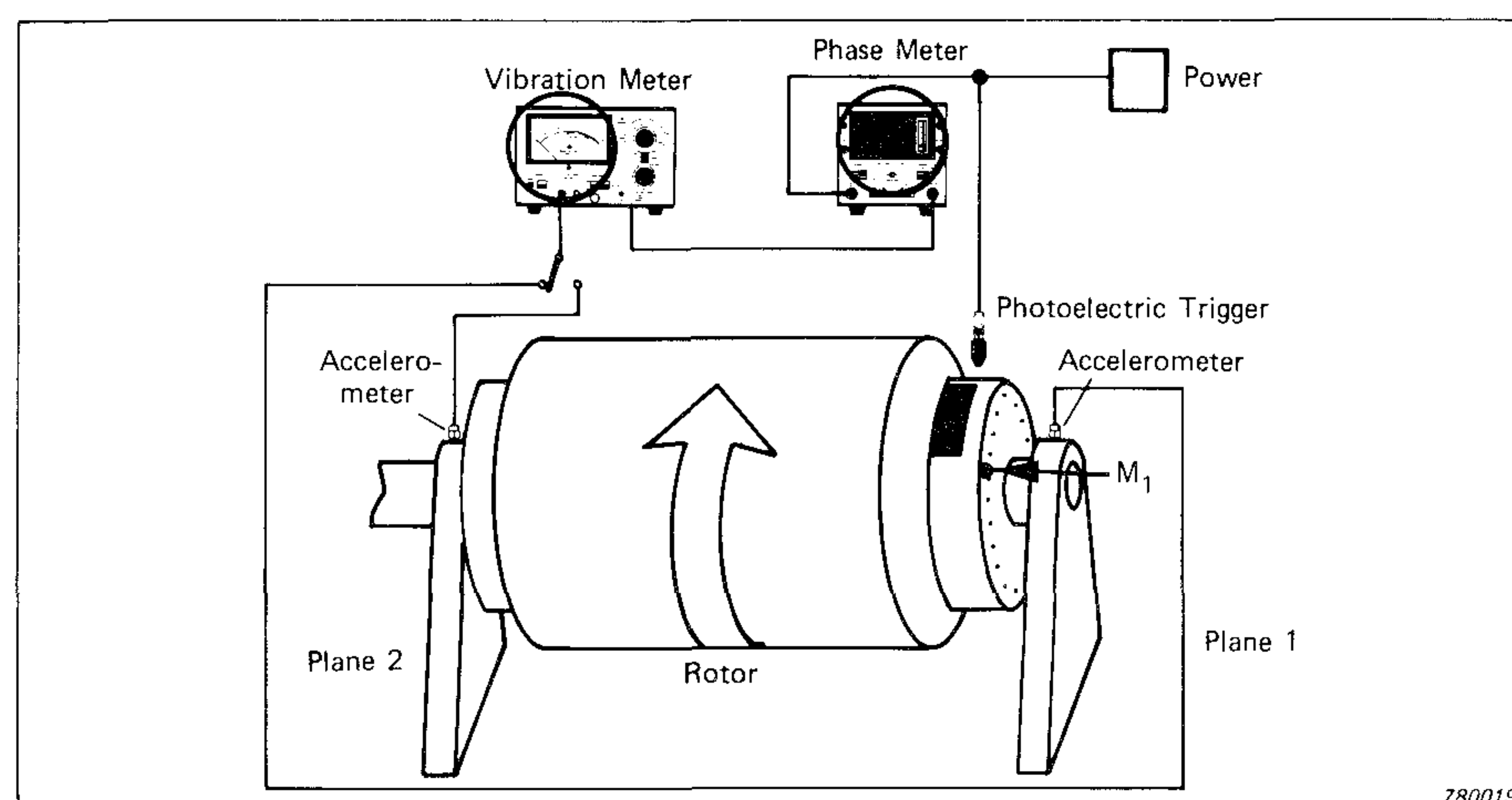
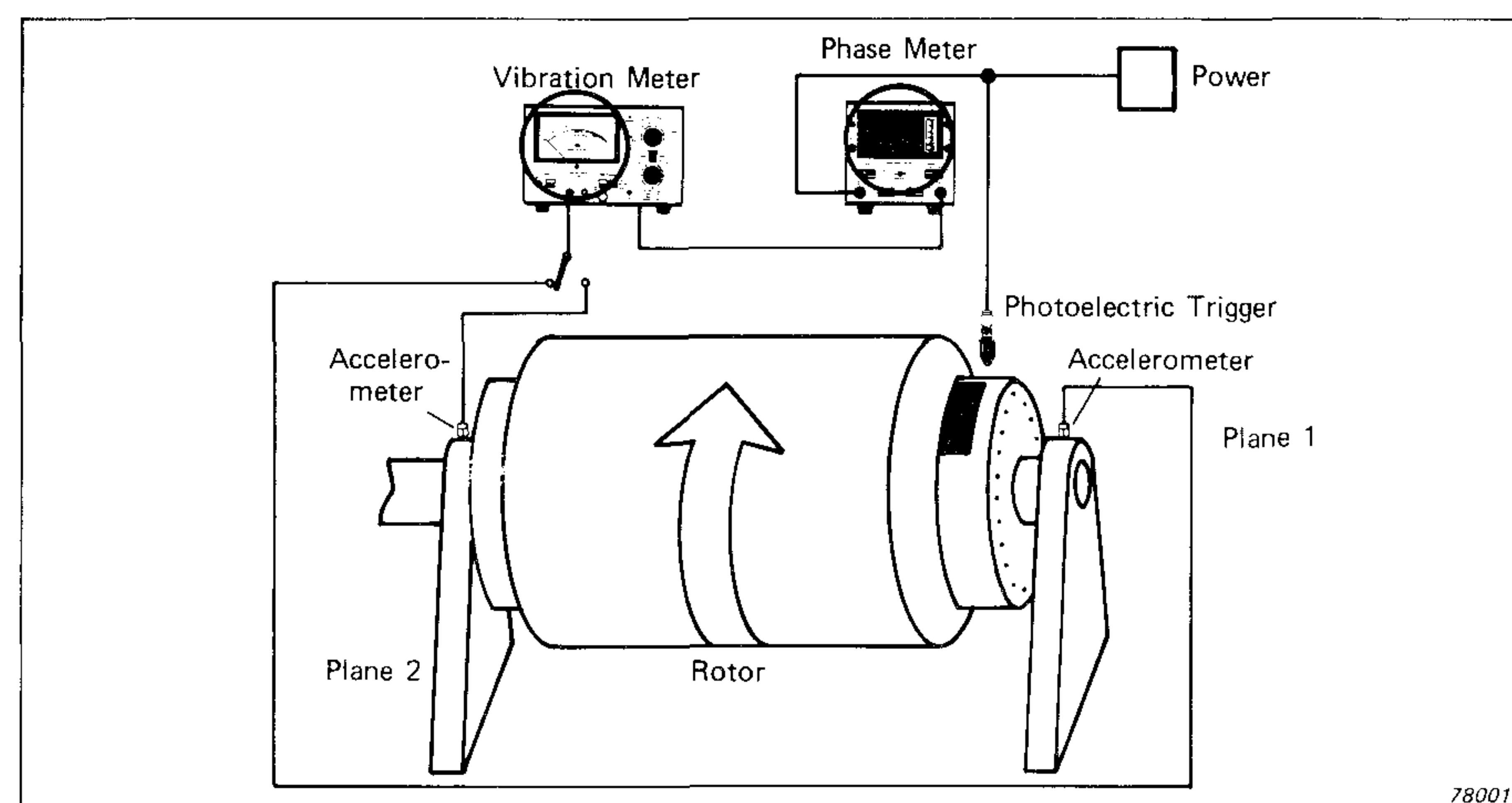
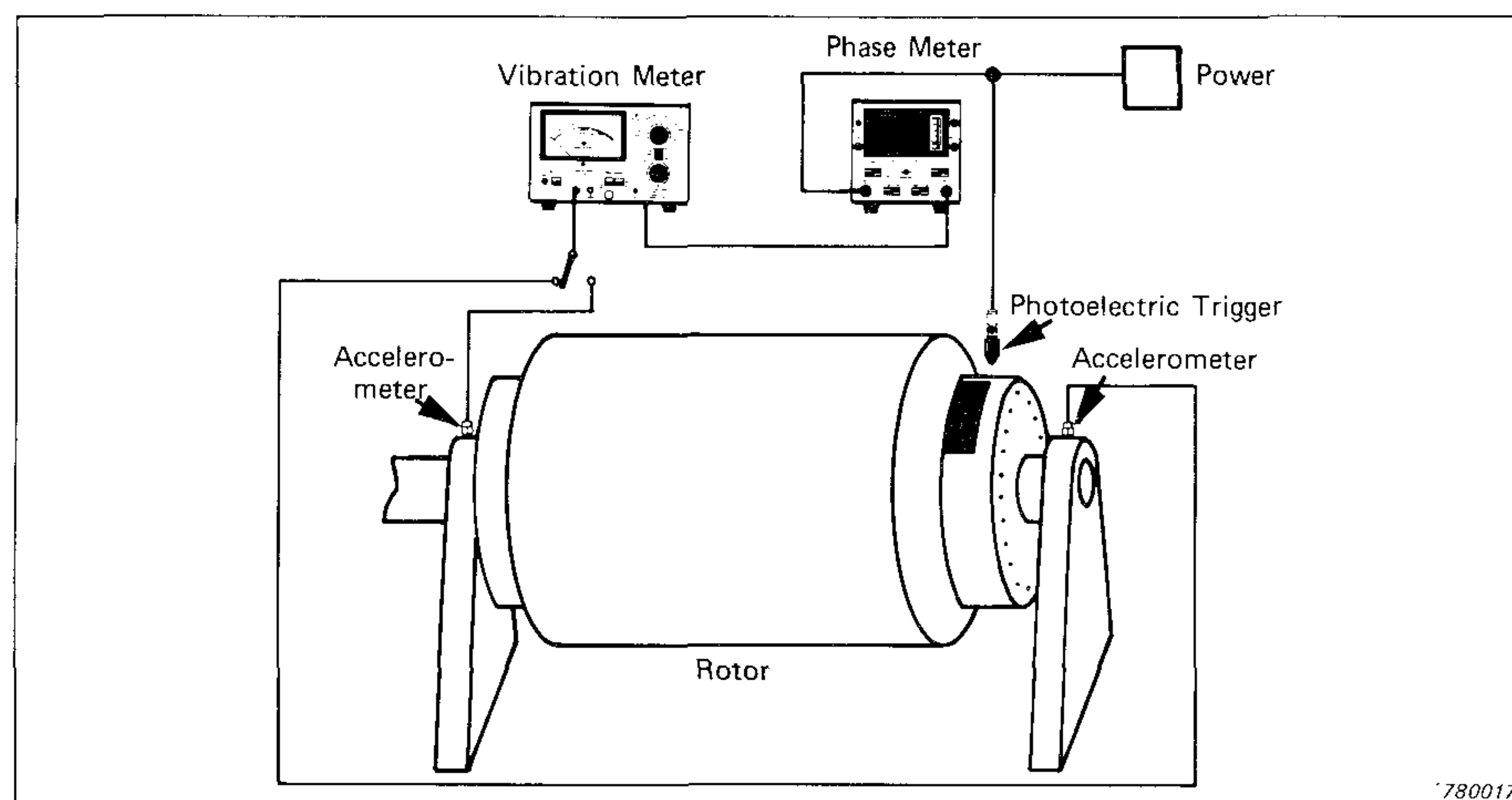
Restart the test rotor.

Note amplitude and phase for Plane 1.

Note amplitude and phase for Plane 2.

Stop the test rotor.

Remove the test mass.



## Trial run 2

Fix a known test mass ( $M_2$ ) onto the rotor at the radius and in the plane where mass correction is to be made, nearest to Plane 2.

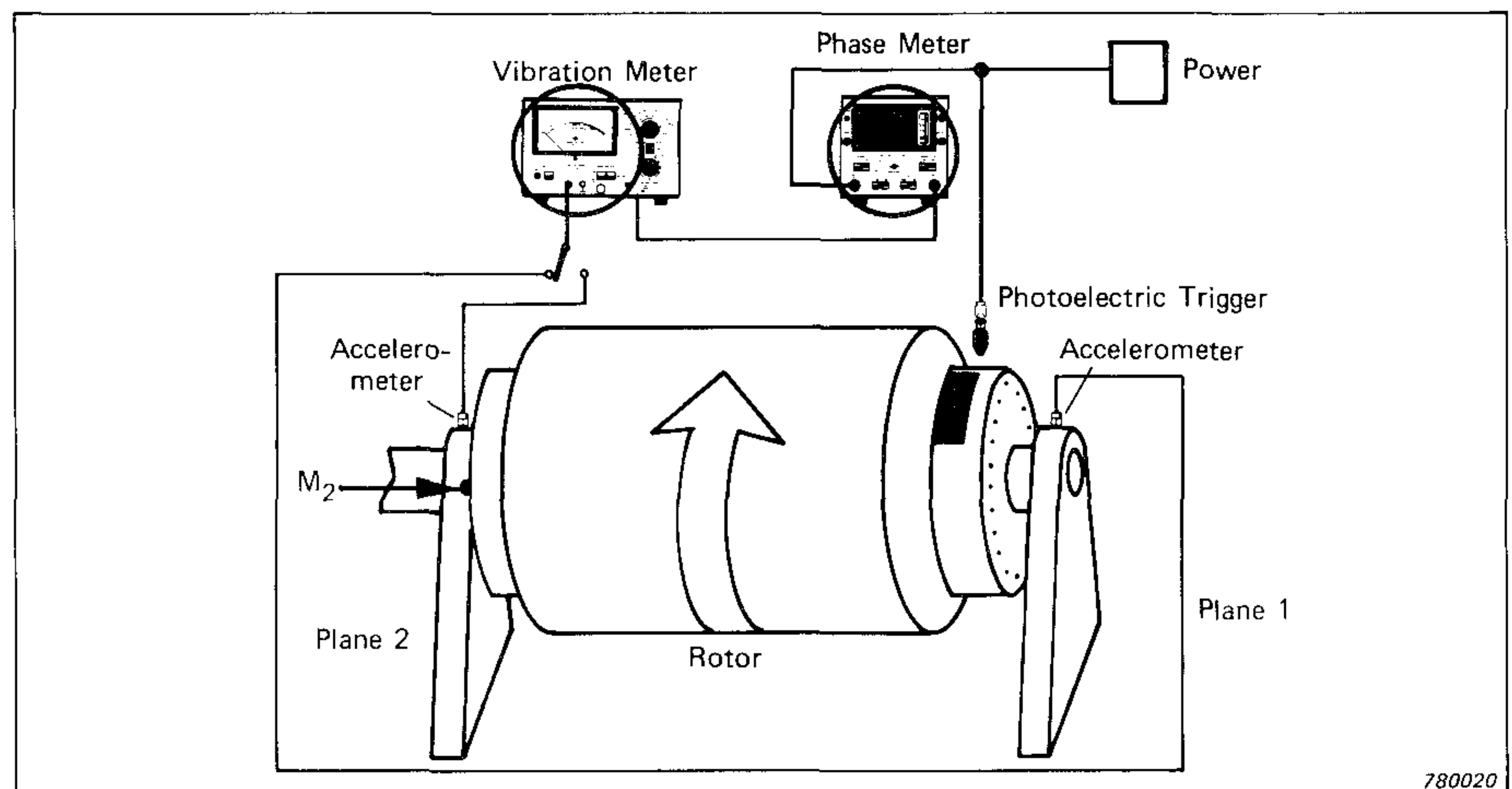
Restart the test rotor.

Note amplitude and phase for Plane 1.

Note amplitude and phase for Plane 2.

Stop the test rotor.

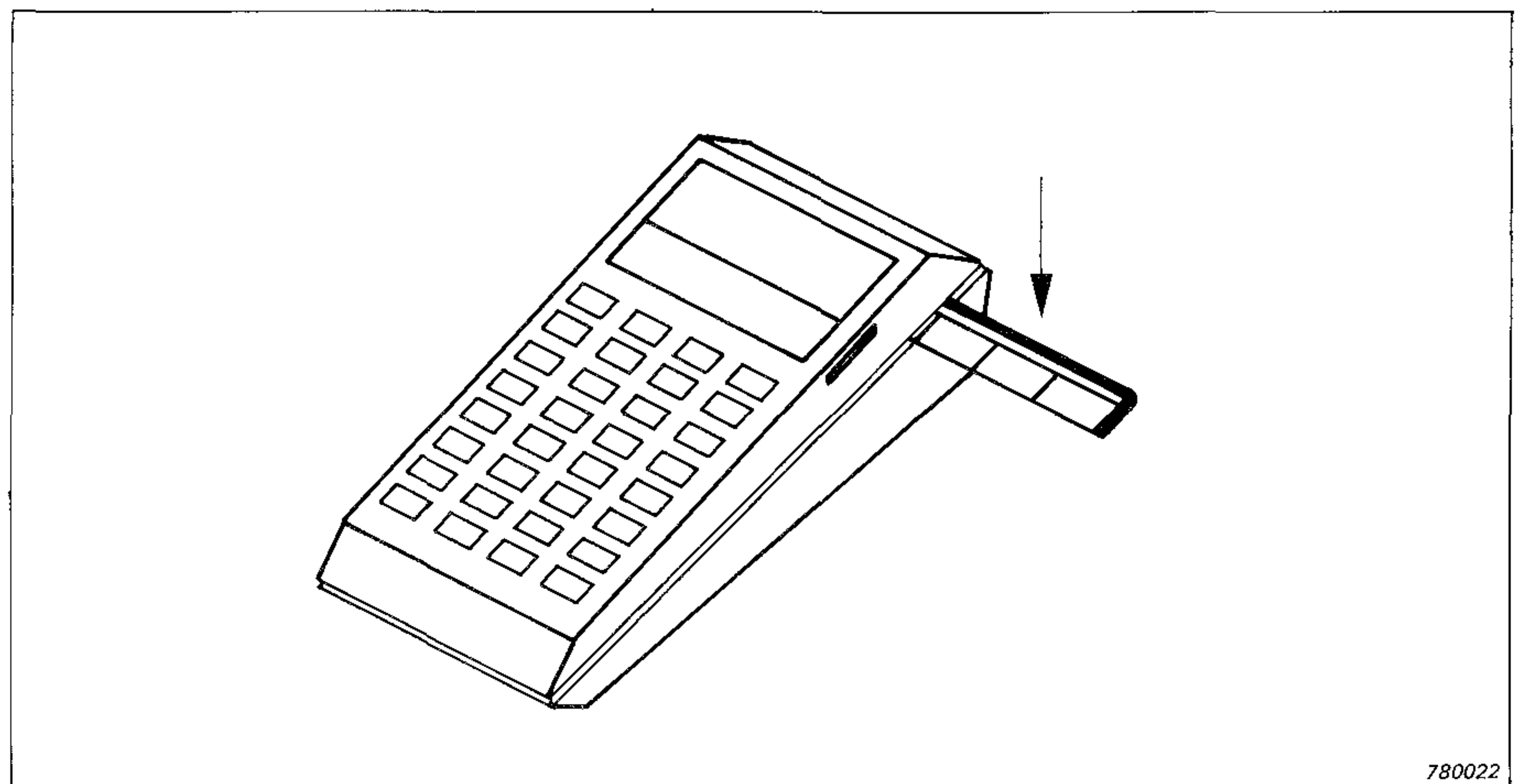
Remove the test mass.



## Calculation

Enter the six values measured for the two planes into a pocket calculator that has been programmed with magnetic cards.

The calculator will now give the correction masses for Planes 1 and 2, plus the angles at which the masses must be attached.

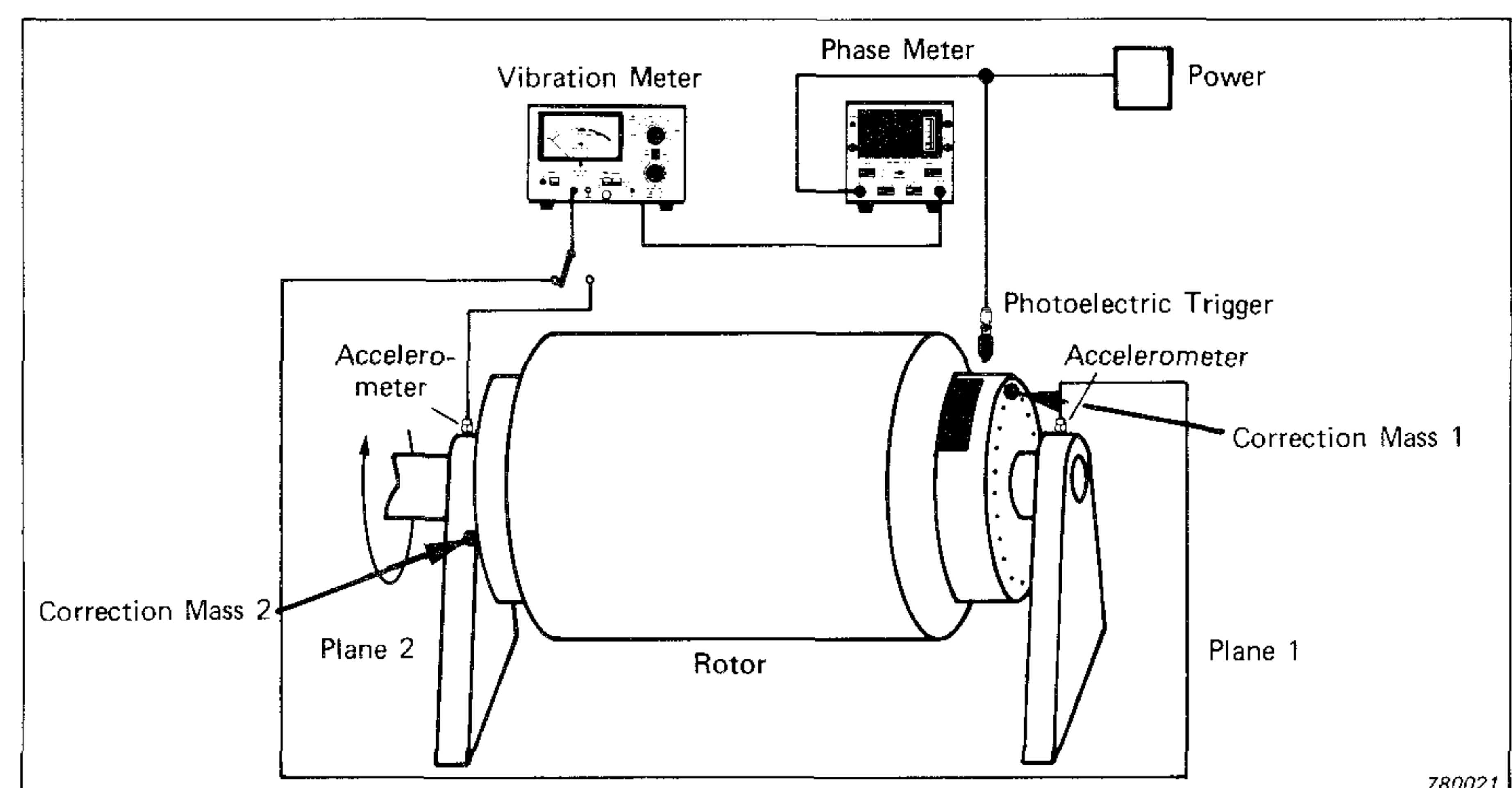


## Correction

Install the calculated correction masses at the calculated angles.

When suitable instrumentation is employed, the whole measurement and calculation procedure need take **only three minutes**.

If sufficiently accurate instrumentation is used (5% : 1°), a repetition of the balancing procedure to achieve a finer balance should be unnecessary.





# Introduction

This Application Note will demonstrate with the aid of several worked examples, how easy it is to balance rotating machinery. Straight-forward methods will be presented that make use of simple portable B & K instrumentation to measure on rotating parts running in their own bearings at normal operating speeds. B & K machines that accept rotating parts and display balancing masses and positions immediately are described in separate publications on the Balancing Machines Type 3905 and Type 3906.

Standards of balance achieved by the arrangements shown here compare favourably with the results obtained from far more complicated and expensive balancing machines.

## Definitions

**Primary Balancing** describes the process where primary forces caused by unbalanced mass components in a rotating object may be resolved into one plane and balanced by adding a mass in that plane only. As the object would now be completely balanced in the static condition (but not necessarily in dynamic) this is often known as **Static Balancing**.

**Secondary Balancing** describes the process where primary forces and secondary force couples caused by unbalanced mass components in a rotating object may be resolved into two (or more) planes and balanced by adding mass increments in those planes. This balancing process is often known as **Dynamic Balancing** because the unbalance only becomes apparent when the object is rotating. After being balanced dynamically, the object would be completely balanced in both static and dynamic conditions.

The difference between static balance and dynamic balance is illustrated in Fig.1. It will be observed that when the rotor is stationary (static) the end masses may balance each other. However, when rotating (dynamic) a strong unbalance will be experienced.

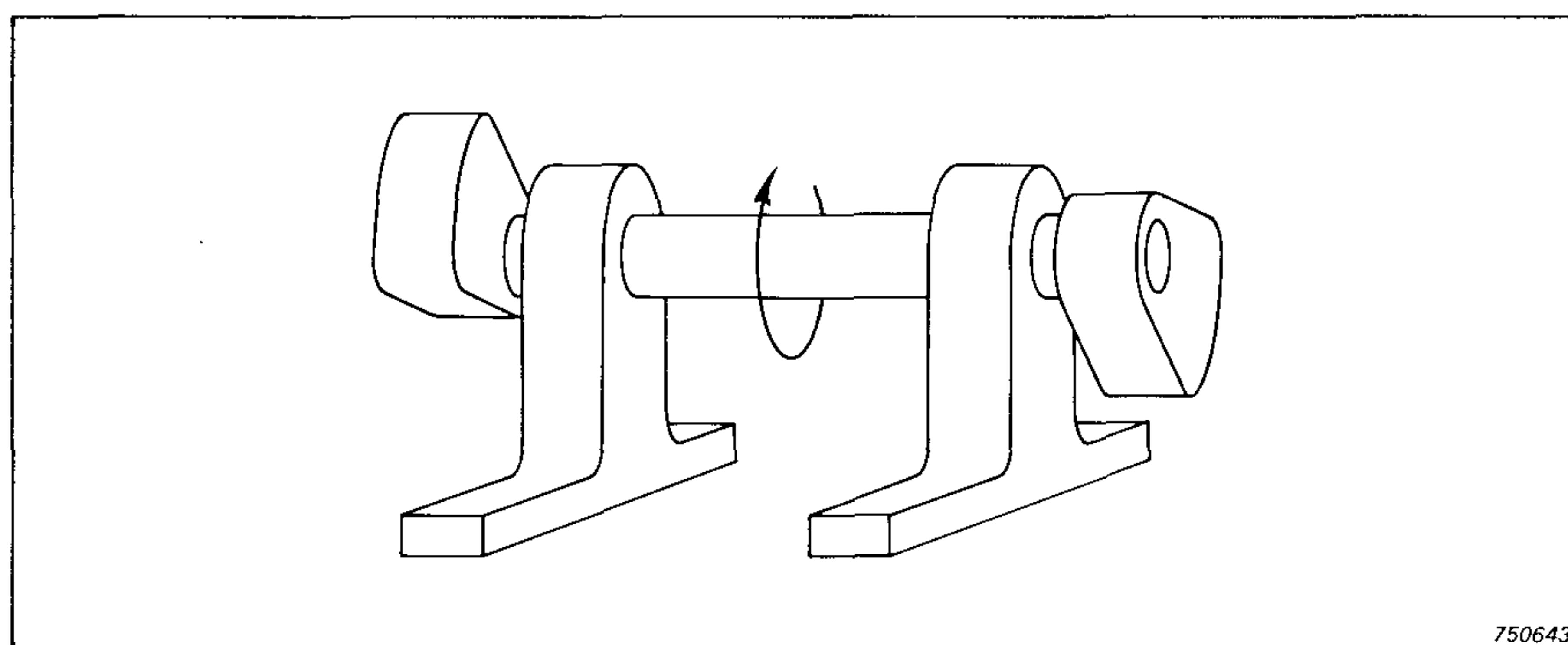


Fig.1. Static Balance, Dynamic Unbalance

## Basic Theory

An object that imparts a vibration to its bearings when it rotates is defined as "unbalanced". The bearing vibration is produced by the interaction of any unbalanced mass components present with the radial acceleration due to rotation which together generate a centrifugal force. As the mass components are rotating, the force rotates too and tries to move the object in its bearings along the line of action of the force. Hence any point on the bearing will experience a fluctuating force. In practice the force at a bearing will be made up from a primary force due to unbalanced mass components in or near to the plane of the bearing, and a secondary force due to unbalanced couple components from the other planes.

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- Foreword
- Introduction
- General Measurement Methods
- Static Balancing Examples
- Dynamic Balancing Measurement
- Dynamic Balancing Calculation
- Appendix 1, Calculator Program for Texas SR-52
- Appendix 2, Mathematical Analysis of Two Plane Balancing

If an accelerometer is mounted on the bearing housing, the fluctuating vibration force can be detected, and an electrical signal sent to a vibration meter. The indicated vibration level is directly proportional to the resultant of the unbalanced masses. The direction in which this resultant acts (i.e. the radius containing the centrifugal force) can be determined in an accurate way by comparing the phase of the fluctuating signal leaving the vibration meter with a standard periodic signal obtained from some datum position on the rotating object.

It is now possible to define the unbalance at the bearing by means of a vector, whose length is given by the magnitude of the unbalanced force (the measured vibration level), and whose angle is given by the direction of action of the force. Further, if the resultant unbalanced force at a bearing can be resolved into its primary (first order moments) and secondary (second order moments) components, it will be possible to balance the object.

Many rotating parts which have most of their mass concentrated in or very near one plane, such as flywheels, grindstones, car wheels, etc., can be treated purely as static balancing problems. This greatly simplifies the calculations, as only the primary components need be taken into consideration. All secondary components can be assumed to be zero.

# General Measurement Methods

## Practical considerations

The rotor to be balanced should be easily accessible, and should have provision for mounting trial masses at various angles around it. The mounting points should preferably be at the same radius from the axis of rotation to simplify calculation. The datum position must be marked on the rotor in such a way that it triggers a pick-up installed alongside it on a stationary part of the machine. A non-contacting type of pick-up is recommended, because this will cause a minimum of disturbance to the rotor.

Vibration level can be measured in terms of acceleration, velocity, or displacement. However as most standards for balancing are written in velocity terms, a legacy of the days when vibration was measured by mechanical velocity sensitive transducers, usually velocity will be the chosen parameter. Use of acceleration levels will tend to emphasize higher frequency components, while displacement will emphasize low frequency components.

## Instrumentation

The basic measuring arrangement, shown in Fig.2, consists of an accelerometer, a vibration meter, and a means of determining the angle of the unbalance relative to the datum position. The most effective method of measuring this angle is to use a phase meter as shown, but a stroboscope can also be used (see Fig.3), or the angle can be deduced from the results of several measurements. The Magnetic Transducer MM 0002 emits a pulse each time the High- $\mu$  Disc passes, and thus establishes a datum position on the circumference of the rotor. Similarly the Photoelectric Tachometer Probe MM 0012 can be fixed to scan the rotor to pick up a triggering mark, for example a piece of adhesive tape or a painted patch with (infrared) reflectivity contrasting with the background. A pulse is emitted for each pass of the mark. The Probe must be powered from a 6 to 10 volt DC supply, such as that available in the Tacho Unit Type 5586. The transducer output is fed

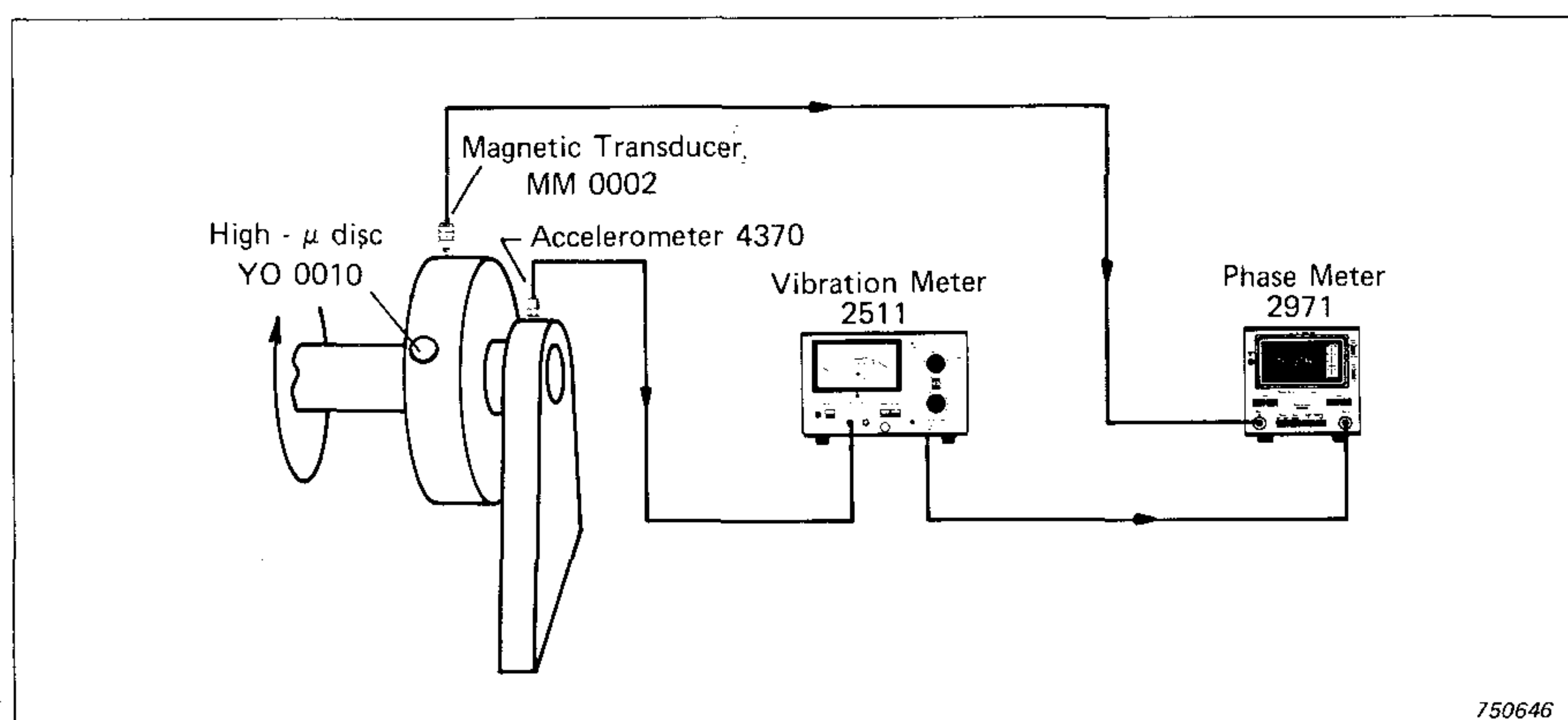


Fig.2. The basic measuring chain

to the reference channel (A) of the Phase Meter Type 2971. The output from the Accelerometer Type 4370 is fed to the Vibration Meter Type 2511 which displays the vibration level. A signal taken from the "Recorder Output" of the Vibration Meter is fed to channel B of the Phase Meter.

When the machine is run, a vibration level will be displayed on the Vibration Meter, and an angle on the Phase Meter, which together give a vector representing the unbalanced mass and its line of action.

Various modifications and additions can be made to the instrumentation arrangement shown in Fig.2 to improve sensitivity, or to take advantage of instruments that are already at hand. Figures 3 to 6, plus 10 to 13 illustrate some of the possibilities.

One very useful addition to the measuring arrangement is the Tunable Band Pass Filter Type 1621, which ensures that the vibration measurements are made at the rotational frequency only, and the Phase Meter (or Stroboscope) receives a clean input signal. Use of the Filter is recommended for installations where the required triggering signal would otherwise be buried in noise, or where high vibration levels occurring at several frequencies cause difficulties in signal tracking. As shown in Fig.5, the Band Pass Filter is connected as external filter to the Vibration Meter, which is actually the same arrangement as in the Vibration Analyzer Type 3513. The Filter has two bandwidths, 3% and 23%, that can be tuned continuously from 0.2 Hz to 20 kHz to match the rotational speed of the machine. It can also be employed to find the relative levels

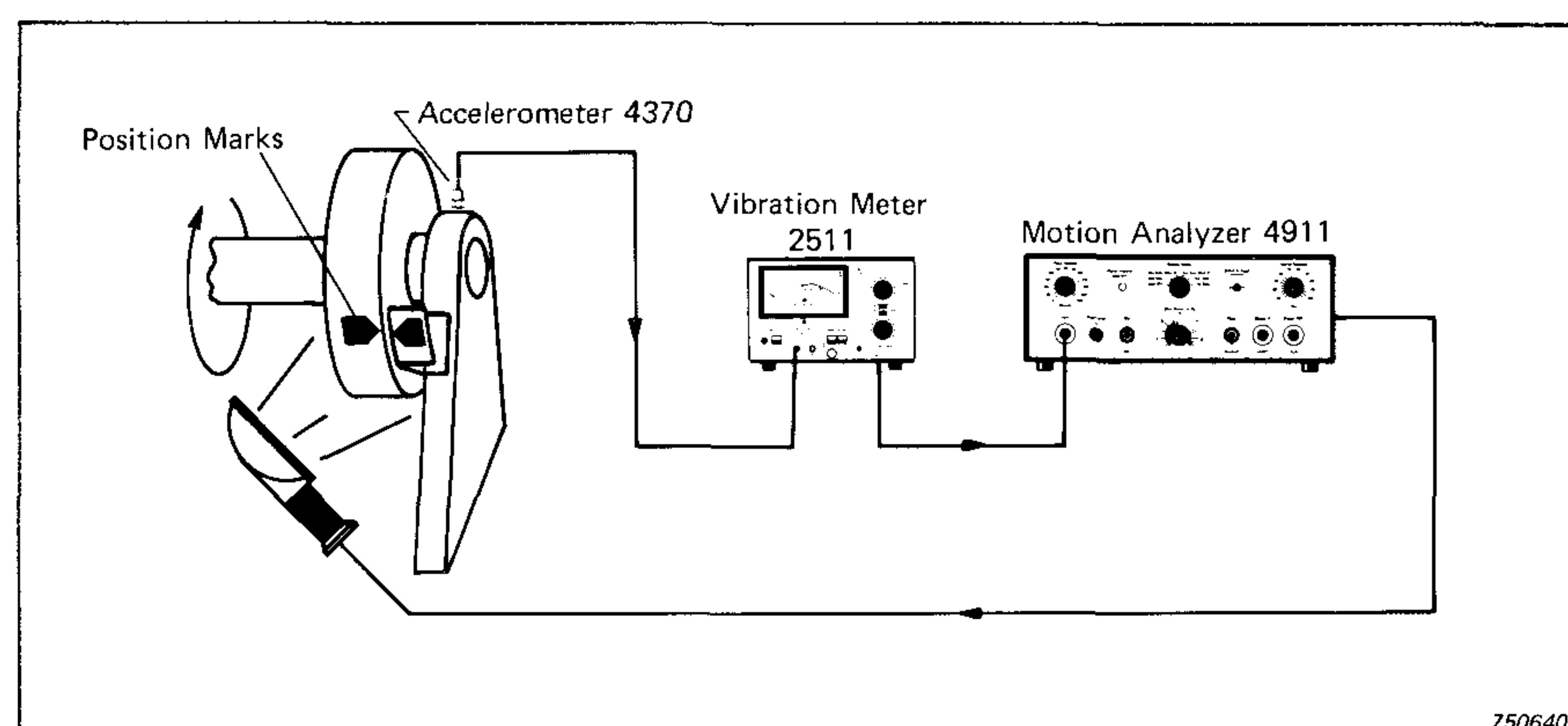


Fig.3. Using a Stroboscope to measure angles



of vibration at the various harmonics, because with some procedures it may be necessary to attempt to balance the harmonic levels too. The Tacho Unit not only powers the Photoelectric Probe, but also gives an indication of the rotational speed.

In operation the Band Pass Filter must be very carefully tuned, because if it is slightly off tune, and the vibration signal falls on the shoulder of the filter curve, phase deviation can be introduced that will falsify Phase Meter readings. One simple way to avoid this effect is to tune the desired frequency as accurately as possible using the 3% bandwidth, and then switch over to the 23% bandwidth to take advantage of the wider peak in the filter characteristic while making the phase measurements.

A more effective alternative which allows the advantage of the 3% pass band with greater external rejection to be maintained while vibration levels and phase angles are being measured uses a change over switch in the connection between the triggering probe and Phase Meter as shown in Fig.6. This switch enables any phase deviation produced by the Band Pass Filter to be eliminated from the phase measurements. It is used as follows.

First set the change over switch to "Reference" so that the trigger signal appears in both channels of the Phase Meter. Set the Phase Meter Slope switches (in opposition) so that the in-phase inputs give a steady reading of  $180^\circ$  or  $3.14$  rad. Run the machine while the Filter is tuned through the rotational frequency until the Phase Meter shows a steady reading of  $180^\circ$  which indicates that the Filter is accurately tuned on the rotational frequency. Set both Phase Meter Slope switches to the same sign, and the change over switch to "Measure" to pass the Accelerometer signal through the now accurately tuned Filter to the Vibration Meter for measurement.

The vibration level is indicated by the Vibration Meter, while the angle measured by the Phase Meter is the actual phase angle of the unbalanced mass referred to a datum.

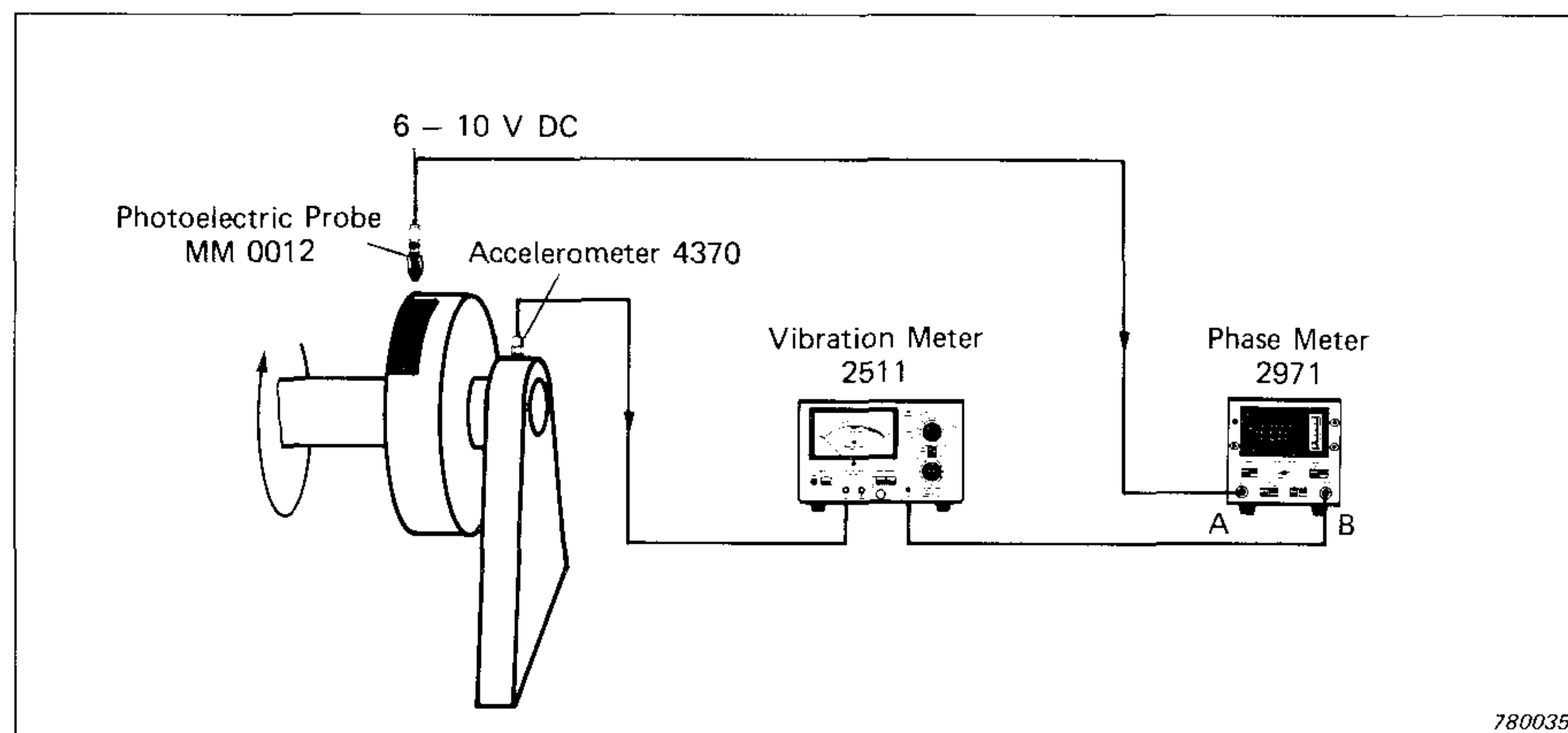


Fig.4. Arrangement with photoelectric triggering

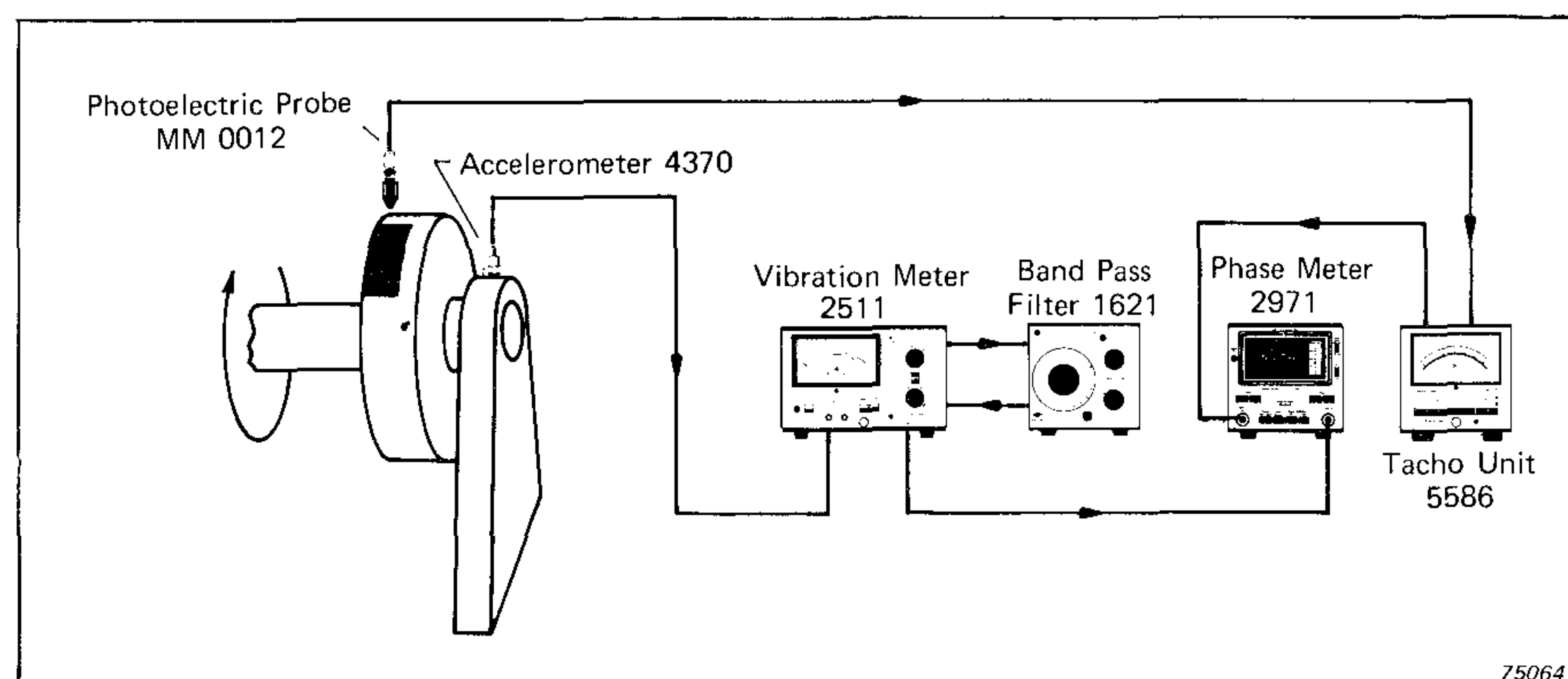


Fig.5. A Band Pass Filter used with the Vibration Meter

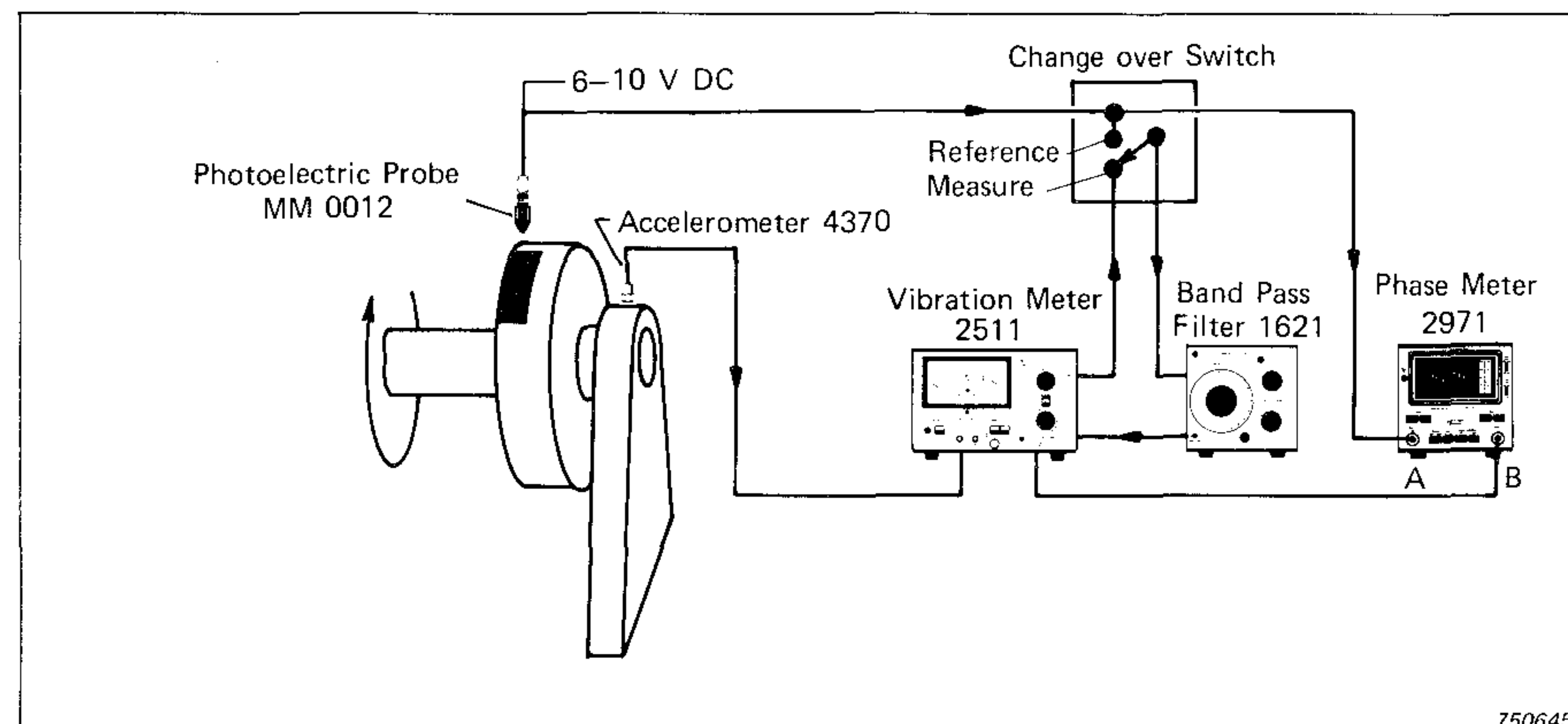


Fig.6. Arrangement with changeover switch

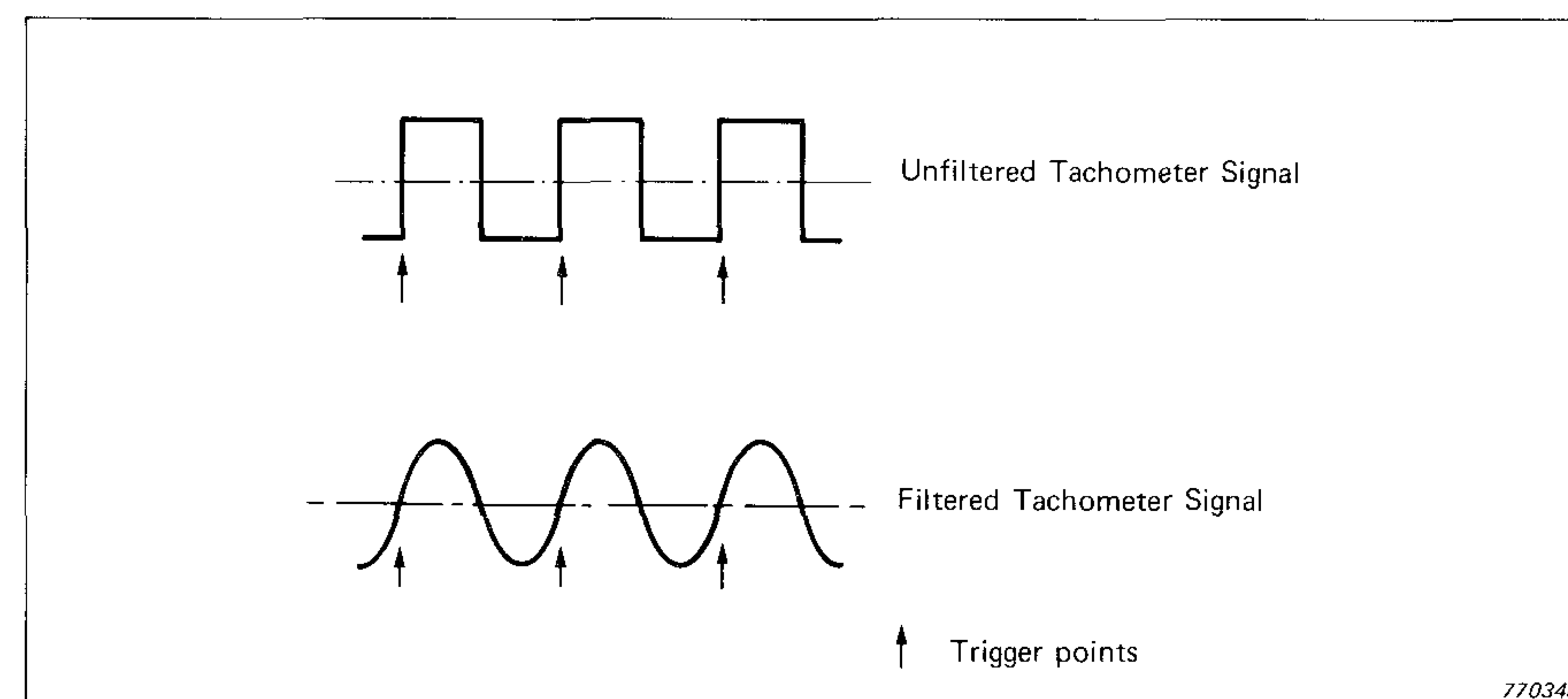


Fig.7. Triggering with a squarewave

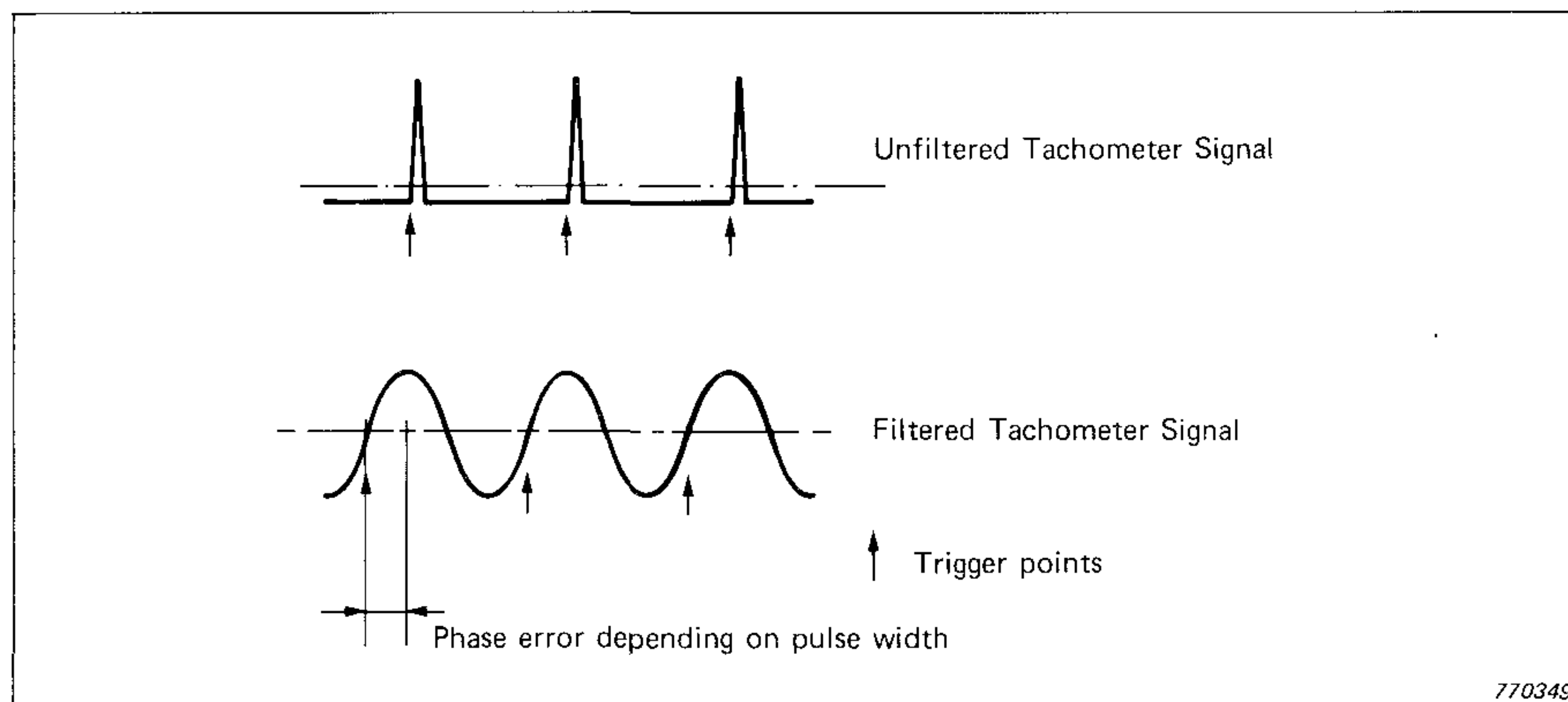


Fig.8. Triggering with a pulse signal

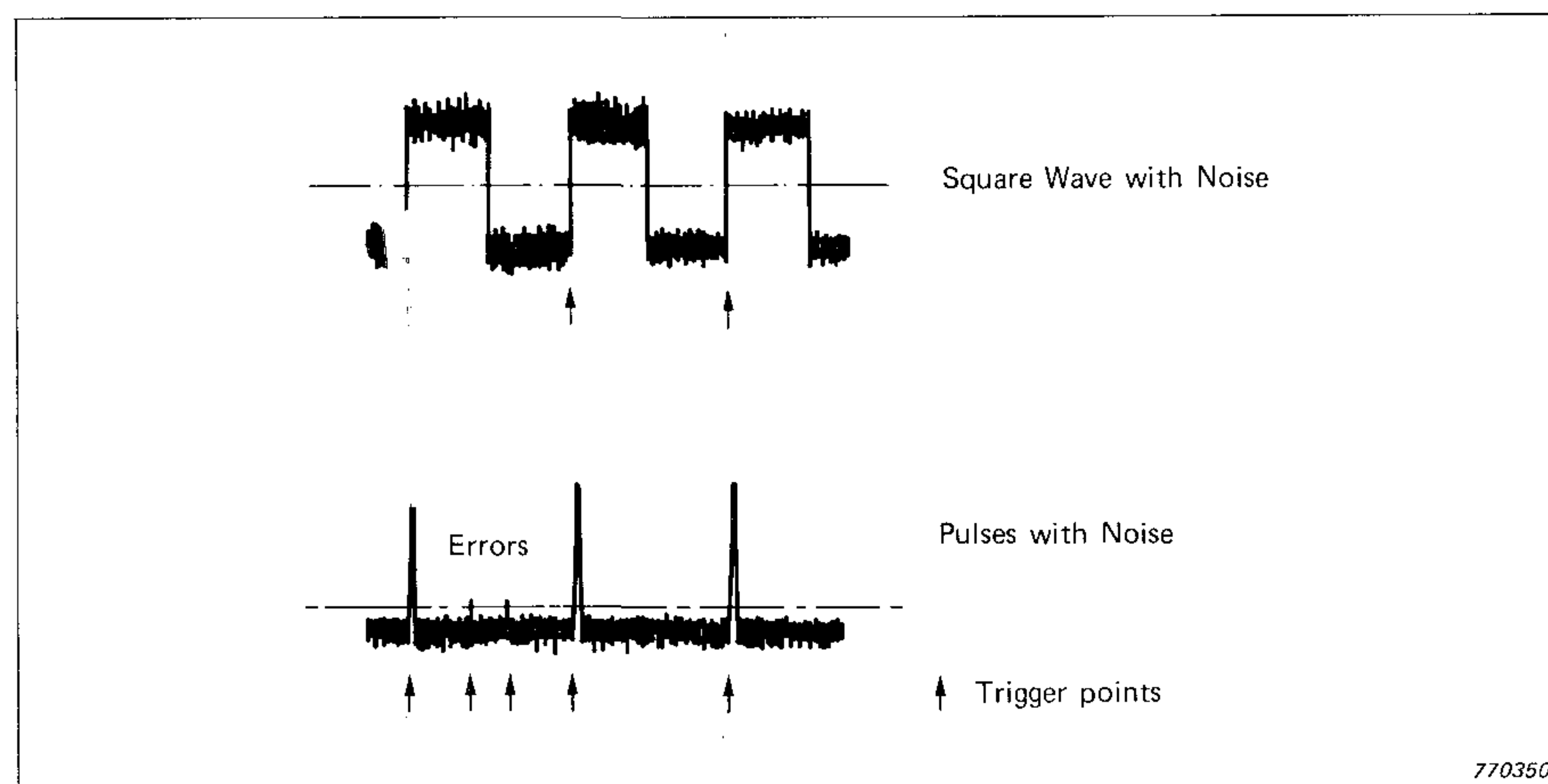


Fig.9. Effects of noise on the trigger signal

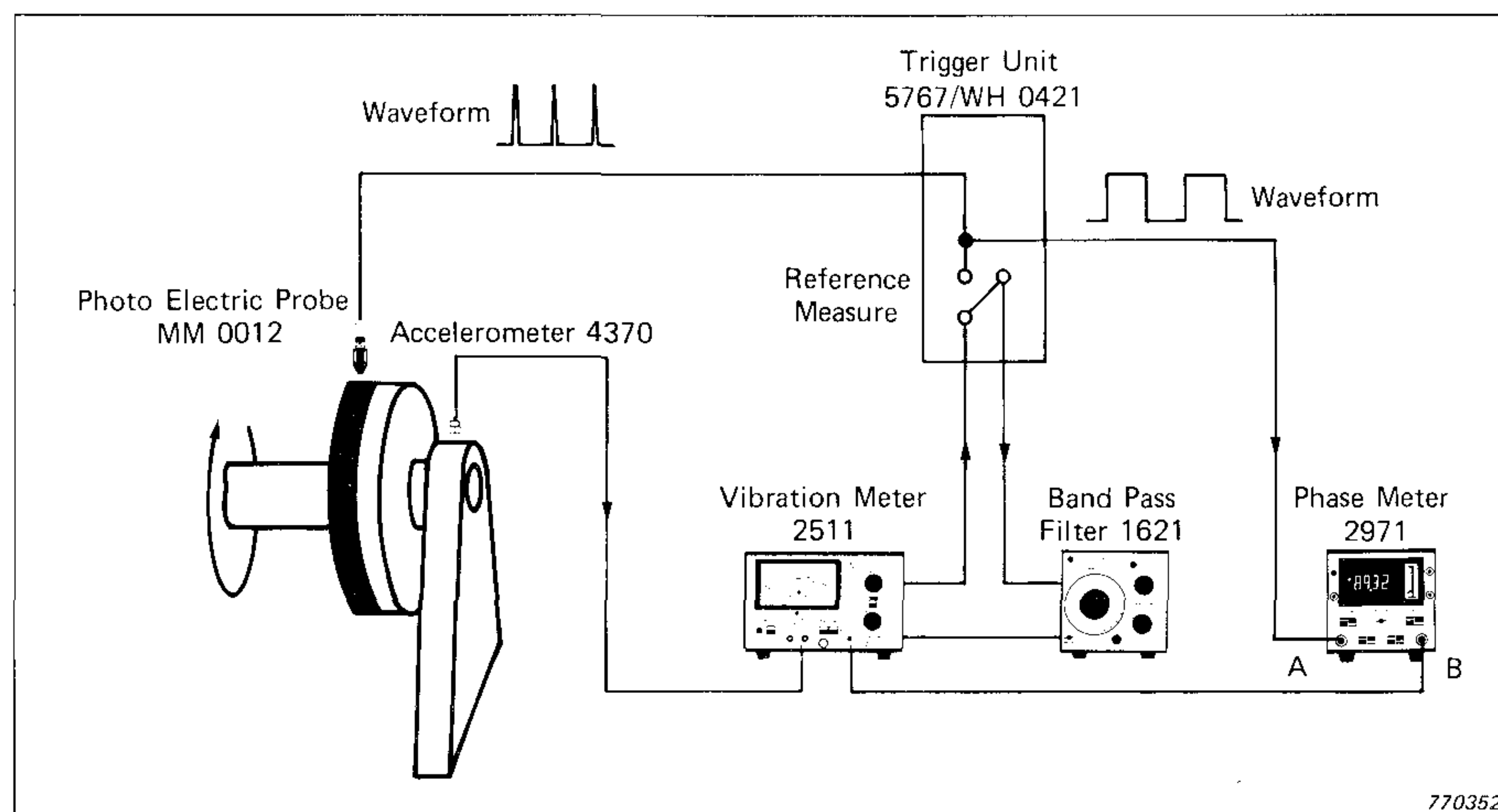


Fig.10. Use of the Trigger Unit

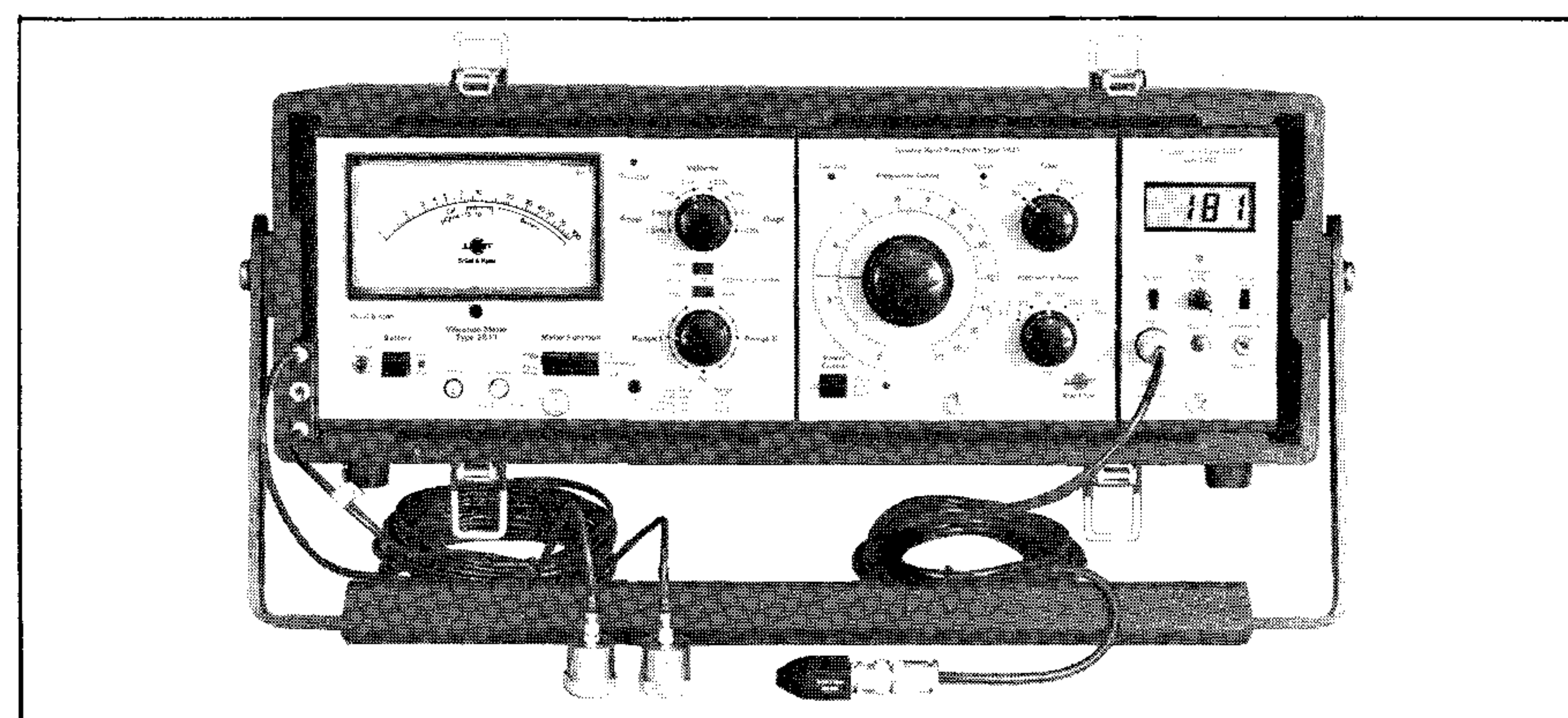


Fig.11. Battery operated Balancing Set Type 3513/WH 0438

It should be noted that this datum is not the same as the plane from which the Phase Meter obtains the triggering signal, because the Phase Meter is triggered by a zero crossing, while the Vibration Meter measures RMS or peak-to-peak. In the calculations, the triggering signal will be the datum used.

For best possible operation of the Phase Meter, the triggering signal should be a square wave. This is easily obtained when using the Photoelectric Probe by fixing marking tape over half the circumference of the shaft or disc monitored by the photo probe. This ensures that the zero crossing of the filtered signal will be the same as for the unfiltered signal.

The square wave triggering signal has another advantage in that a phase meter operating on zero crossings will not be subject to error, even with a noisy signal, whereas only a small amount of noise on the pulse signal can disturb the measurements.

In consequence, one of the Trigger Units Type 5767 (either WH 0421 or WH 0422) can be usefully employed together with the MM 0012, to produce a symmetrical square wave signal when triggered by any waveform having a frequency between 1 Hz and 4 kHz. Both versions feature a trigger error indicator, and an adjustable trigger level to allow the triggering points to be lifted above the noise level. A change-over switch like that shown in Fig.6 is also included in both versions, together with a power supply for the Photoelectric Probe. The main item that distinguishes the WH 0422 is its built-in phase meter able to measure the angle between the vibration signal and the datum. The result is expressed in degrees which are shown on a digital display panel.

A measuring arrangement employing a Trigger Unit Type 5767/WH 0421 and separate Phase Meter together with the Vibration Analyzer Type 3513 is shown in Fig.10. The similar arrangement seen in Fig.11 that uses only battery operated equipment is available as Balancing Set Type 3513/WH 0438. This set is particularly convenient to use be-



cause all the instruments can be installed in the carrying case, and being battery powered the Balancing Set can be carried and operated practically anywhere. The electrical connections are the same as in Fig.10, with the Trigger Unit and Phase Meter combined in the Trigger Unit 5767/WH 0422. The Balancing Set Type 5786/WH 0439 is another fully portable arrangement that has all its battery powered in-

struments installed in one carrying case. This set is available as an economical alternative that uses an instrument arrangement more specifically intended for balancing measurement.

The greatly simplified measuring arrangement shown in Fig.13 can also yield results that can be used in balancing a rotor. The procedure is not quite the same as for the

other measuring arrangements, it will be explained in Example 3. As the methods used with the other arrangements are all basically similar to each other, worked examples will not be presented for each arrangement.

All the measuring arrangements described and illustrated here are equally suitable for both static and dynamic balancing.

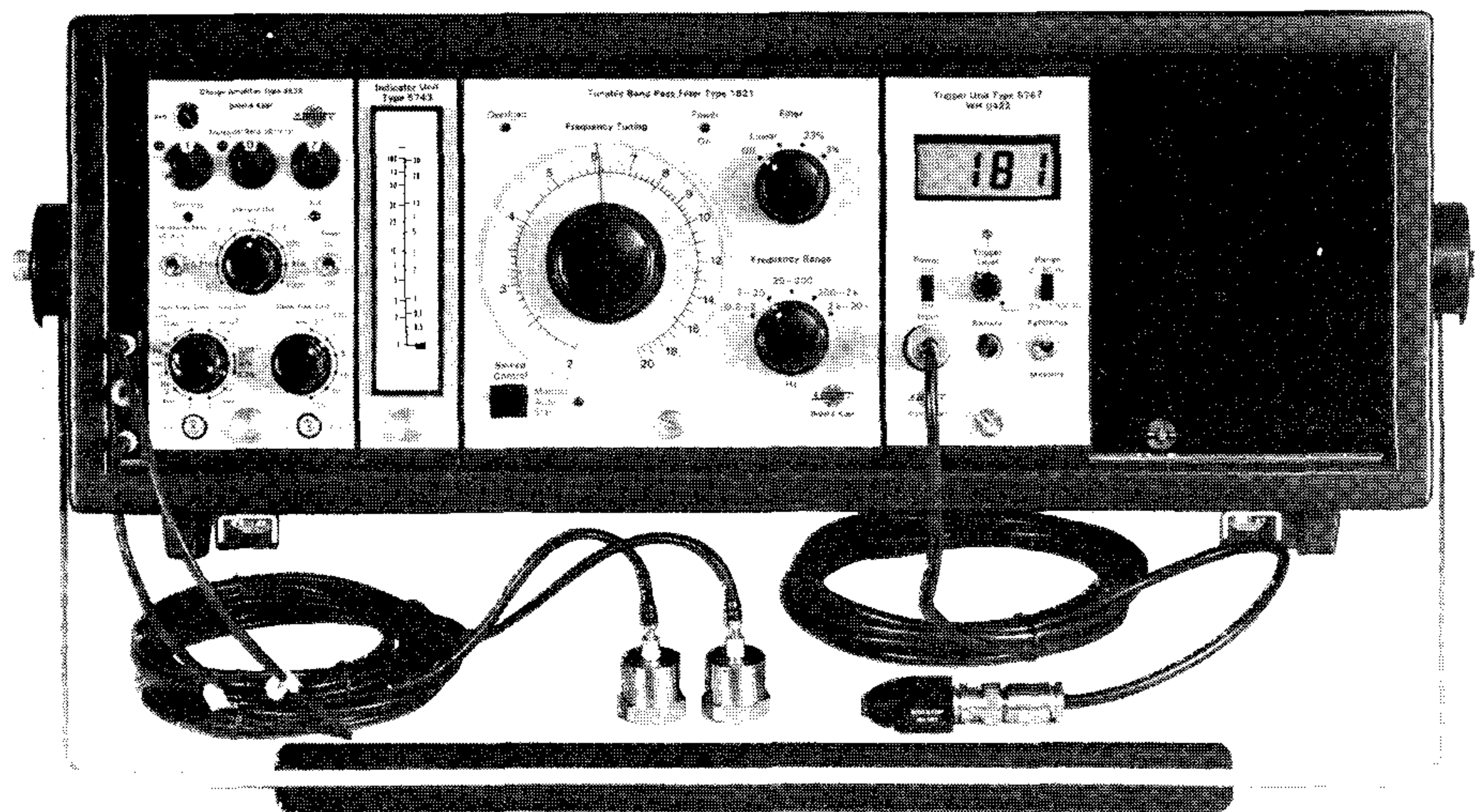


Fig.12. Balancing Set Type 5786/WH 0439

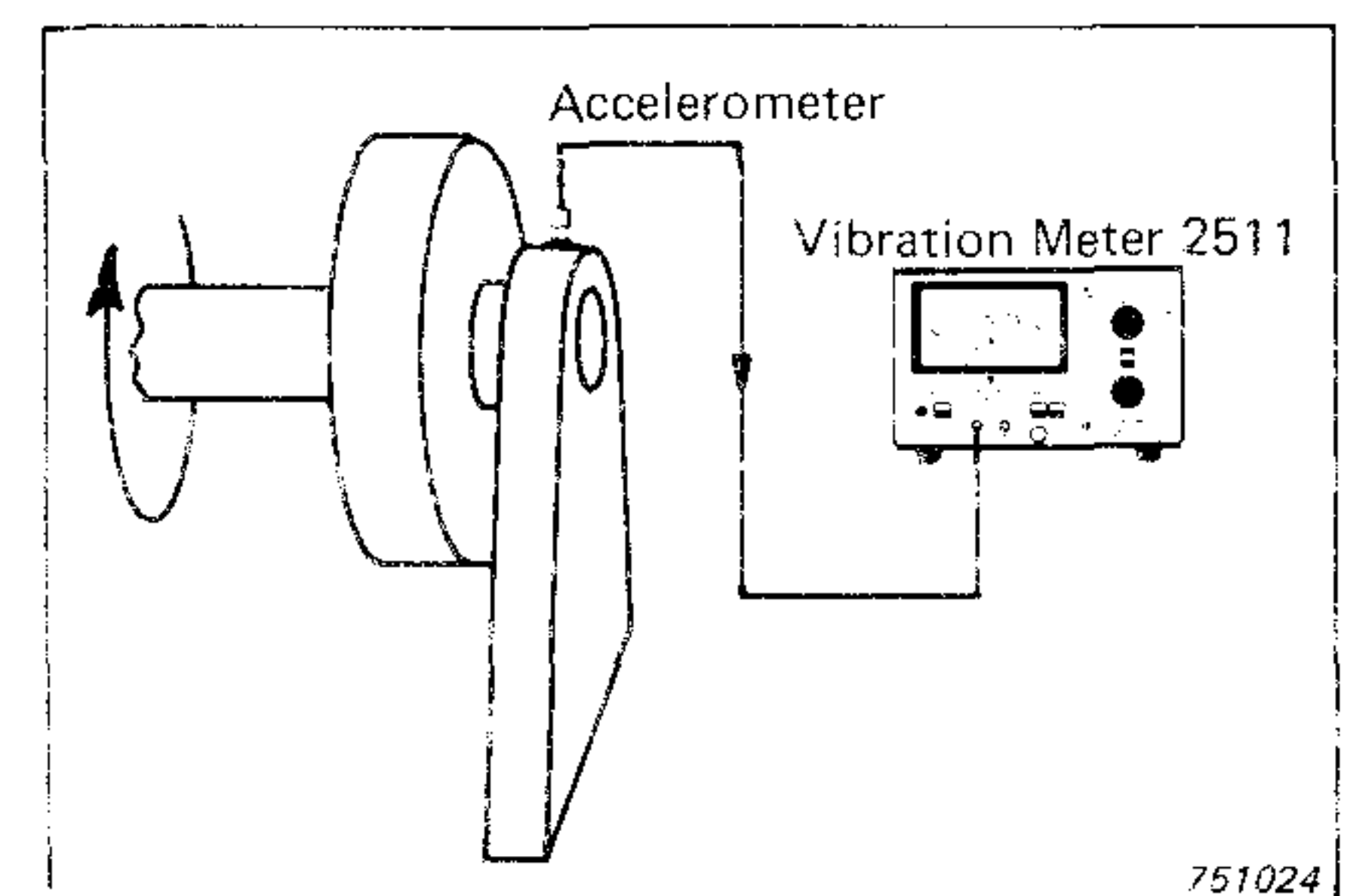


Fig.13. Simplified arrangement using only a Vibration Meter and Accelerometer

## Static Balancing; Measurement and Calculation

### Example 1,

*To balance a rotor statically by using a Motion Analyzer Type 4911 to measure phase angles with the arrangement shown in Fig.3.*

The Vibration Meter was switched to measure vibration "Velocity", to obtain equal emphasis of both low and high frequencies, the internal filters in the Vibration Meter were used to restrict the measuring range to frequencies between 10 Hz and 1000 Hz, to improve the signal to noise ratio. Peak-to-peak measurement with one second time constant was chosen so that a large and responsive needle deflection could be obtained. An AC signal from the "Recorder Output" of the Vibration Meter was fed to one of the "Input" terminals of the Motion Analyzer to be used as a triggering signal. The Analyzer was switched to "External Synchronised" mode

so that the lamp would blink at rotation frequency. A position mark was made on the rotor, with another alongside it on the stationary part of the machine.

The machine was run up to its operating speed (2800 rpm), and a vibration velocity level of 15 mm/s indicated by the Meter. Because the flashing of the lamp was synchronized with the rotation, when the machine was observed by the light, the rotor appeared to be stationary. Turning the "Phase Deviation" knob allowed the position mark on the rotor to be lined up with the stationary mark on the machine. The "Phase Deviation" control is graduated in 10° steps so that it was possible to estimate a phase angle of 55°, after which the machine was stopped. Together, the velocity level and phase angle give a vector that

represents the original unbalance of the rotor,  $V_0$  in Fig.14.

A trial weight of known mass was fixed at a known radius at an arbitrary angular position on the rotor. A mass with sufficient magnitude to produce a pronounced effect on the rotor must be used for the first trial. In the example, a trial mass of 5 g was fixed to the rotor at the same angular position as the reference mark. Then the machine was run up to its operating speed again. The new vibration velocity level was 18 mm/s, and when the position marks on the rotor and stationary part of the machine had been lined up with each other by the "Phase deviation" control, the new phase angle was found to be 170°. These values represent the resultant effect of the initial unbalance and the 5 g trial mass, which is shown as vec-



tor  $V_1$  in Fig.14.

Now sufficient information was available for the vector diagram in Fig.14 to be constructed, with vector lengths proportional to the measured vibration velocity levels obtained from the Vibration Meter, and angles being those indicated by the Motion Analyzer. First draw vector  $V_0$ , then as vector  $V_1$  is the resultant of the initial unbalance plus the 5 g mass, vector  $V_T$  can be found, which represents the trial mass alone. The length of  $V_T$  is proportional to the 5 g mass, so that the length of vector  $V_0$  (the initial unbalance) can be determined in mass units. The phase of  $V_T$  gives the angle at which the trial mass was fastened, so that it is a simple matter to determine the angle the initial unbalance makes with the position of the trial mass. Hence the angular position for the compensating mass can easily be found.

The original unbalance is given by

$$M_0 = \frac{V_0}{V_T} \times M_T$$

(With  $V_0/V_T$  a scaling factor)

$$= \frac{15}{29} \times 5 = 2,6 \text{ g}$$

So the compensating mass

$$M_{\text{COMP}} = 2,6 \text{ g}$$

And its position is given by

$$\begin{aligned} \angle_{\text{COMP}} &= -\angle_T + \angle_0 + 180^\circ \\ &= -198^\circ + 55^\circ + 180^\circ \\ &= +37^\circ \text{ referred to the} \\ &\text{position of the Trial Mass} \end{aligned}$$

The positive angle means that the compensating mass is to be fastened at  $37^\circ$  from the trial mass position in a positive direction, that is, in the direction of rotation.

After the rotor has been balanced by this procedure, it is recommended that the vibration level be measured again to check the standard of balancing obtained. Due to non-linearities or inaccuracies in the practical measuring arrangement, the rotor may not have been sufficiently well balanced by one application of the balancing procedure. When the level of the residual unbalance is unacceptably high, the whole balancing procedure must be

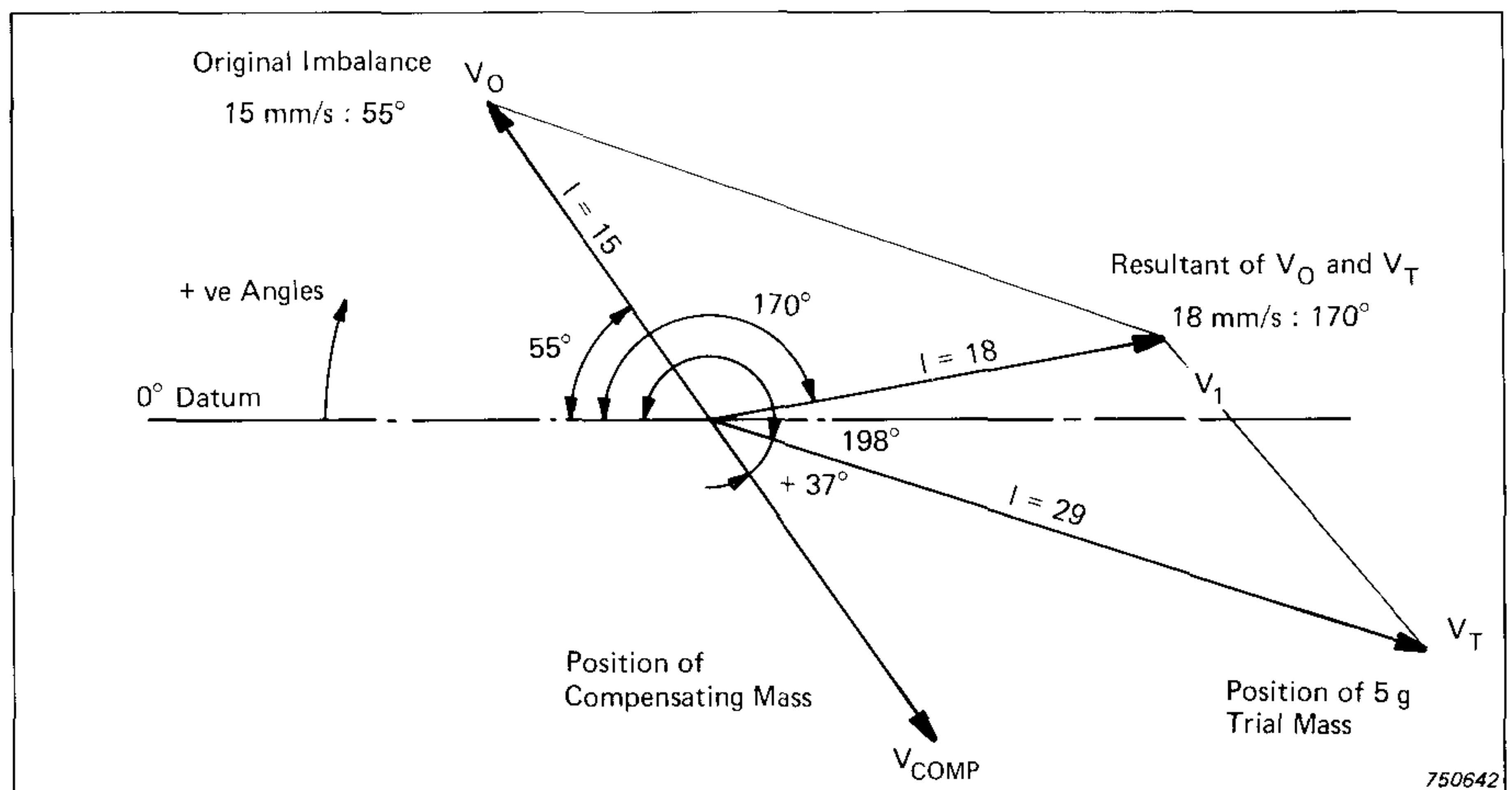


Fig.14. Vector diagram for Example 1

reiterated, until an acceptable level is achieved.

Programs for use with externally programmable calculators, similar to those used for two plane balancing, are available to simplify these graphic procedures.

### Example 2,

*To balance a rotating machine statically by using the arrangement shown in Fig.5 to measure vibration levels and phase angles.*

Measurement of peak-to-peak vibration velocity level was selected

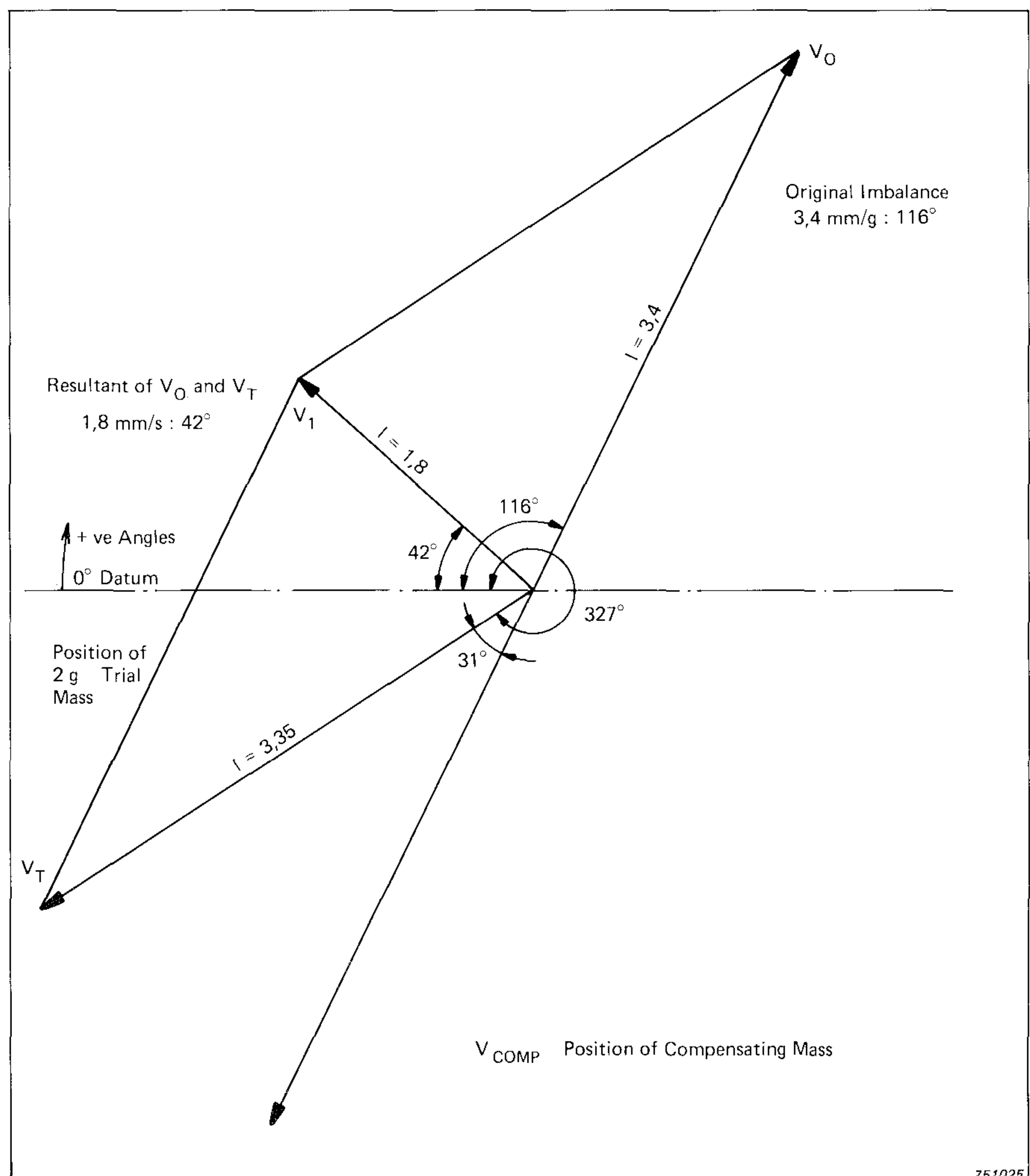


Fig.15. Vector diagram for Example 2



on the Vibration Meter, and a bandwidth of 3% on the Band Pass Filter. The machine was run up to its normal operating speed (1490 rpm on the Tacho Unit), after which the band pass filter centre frequency was adjusted to give the highest indicated level of vibration velocity on the Vibration Meter. A vibration level of 3,4 mm/s was recorded, and when the bandwidth was broadened to 23%, the Phase Meter indicated + 116°.

The machine was stopped, and a 2 g trial mass fixed to it. When the machine was running again, the vibration velocity level was found to have decreased to 1,8 mm/s, while the phase angle had changed to + 42°.

The position and magnitude of the compensating mass were determined from the vector diagram shown in Fig.15.

The original unbalance is given by

$$M_0 = \frac{3,4}{3,35} \times 2 = 2,03 \text{ g}$$

So the compensating mass  
 $\angle_{\text{COMP}} = 2,03 \text{ g}$

And its position is given by  
 $\angle_{\text{COMP}}$

$$\begin{aligned} &= -\angle_T + \angle_0 + 180^\circ \\ &= -327^\circ + 116^\circ + 180^\circ \\ &= -31^\circ \text{ referred to the} \\ &\text{position of the trial mass.} \end{aligned}$$

As the angle indicated is negative, the compensating mass is to be fastened at 31° in the negative direction from the trial mass, which is the opposite direction from the rotation.

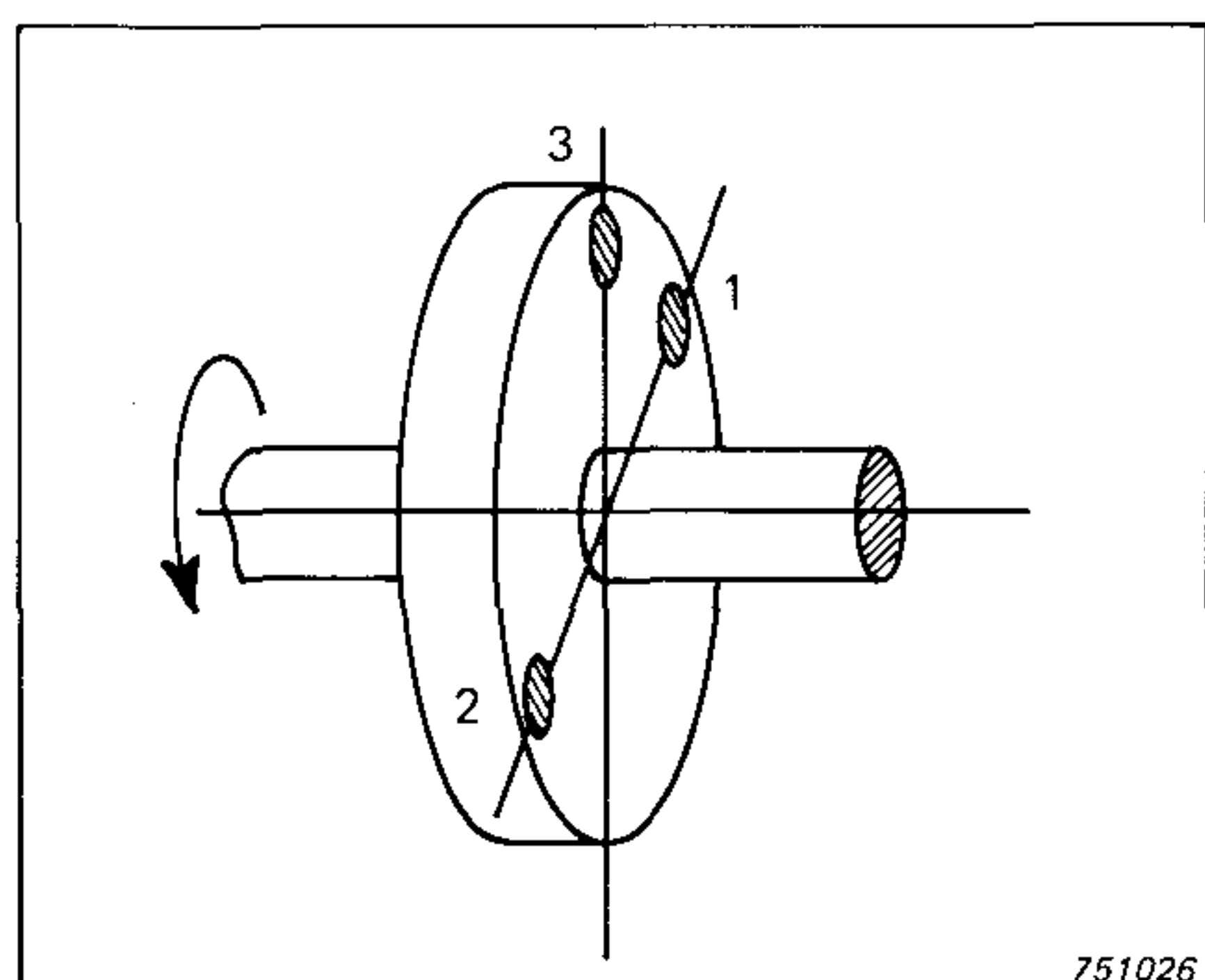


Fig.16. Positions of the trial mass used in Example 3

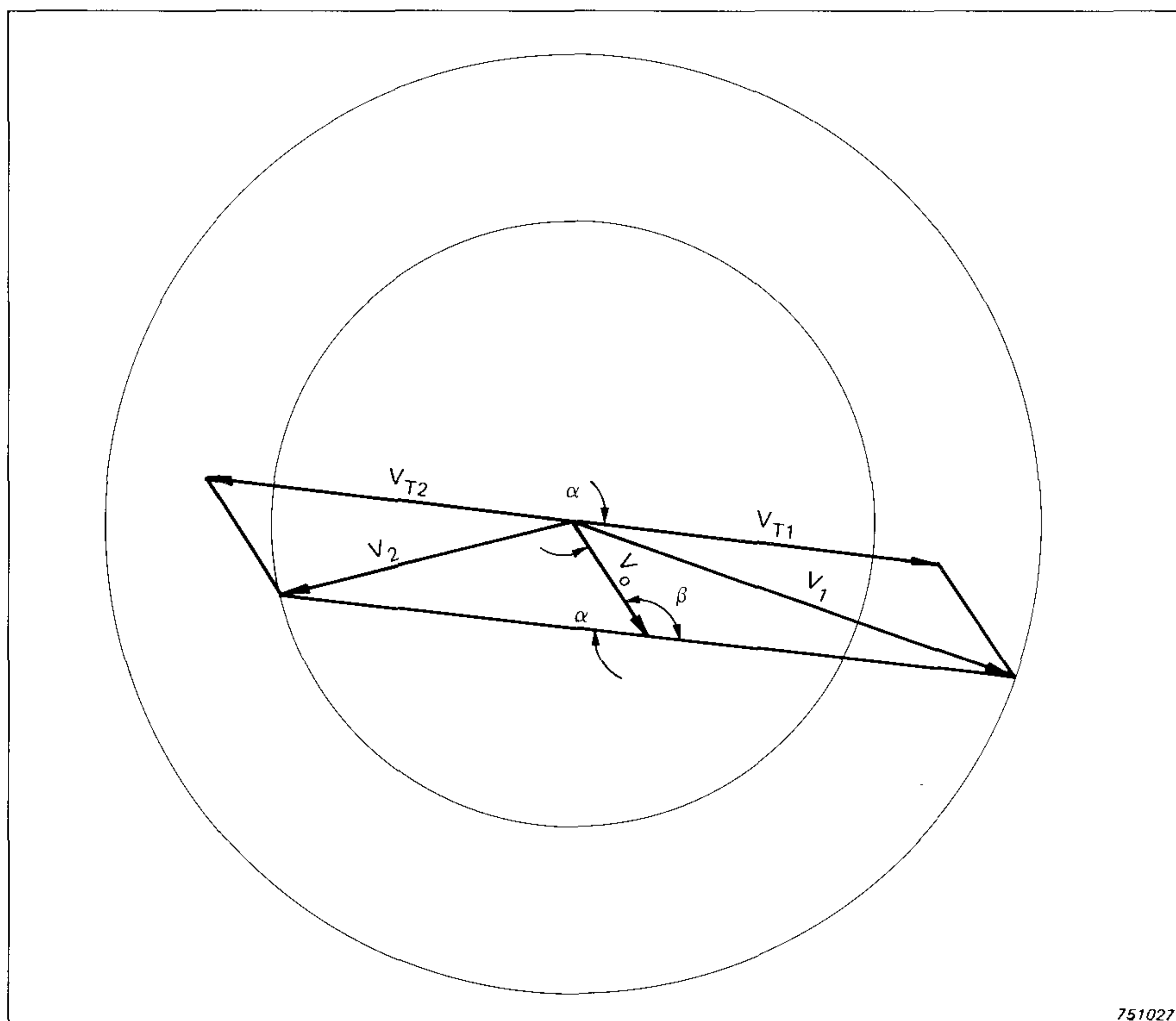


Fig.17. Geometry for Example 3

### Example 3,

*To balance a rotating machine statically, using only a Vibration Meter and an Accelerometer in the arrangement shown in Fig.13.*

With this method the simpler instrumentation employed must be compensated by additional trial running, and four runs are required, each with a vibration measurement to be taken at the bearing. A trial mass will be required which can be mounted at the same radius in three different positions at 90° from each other as shown in Fig.16.

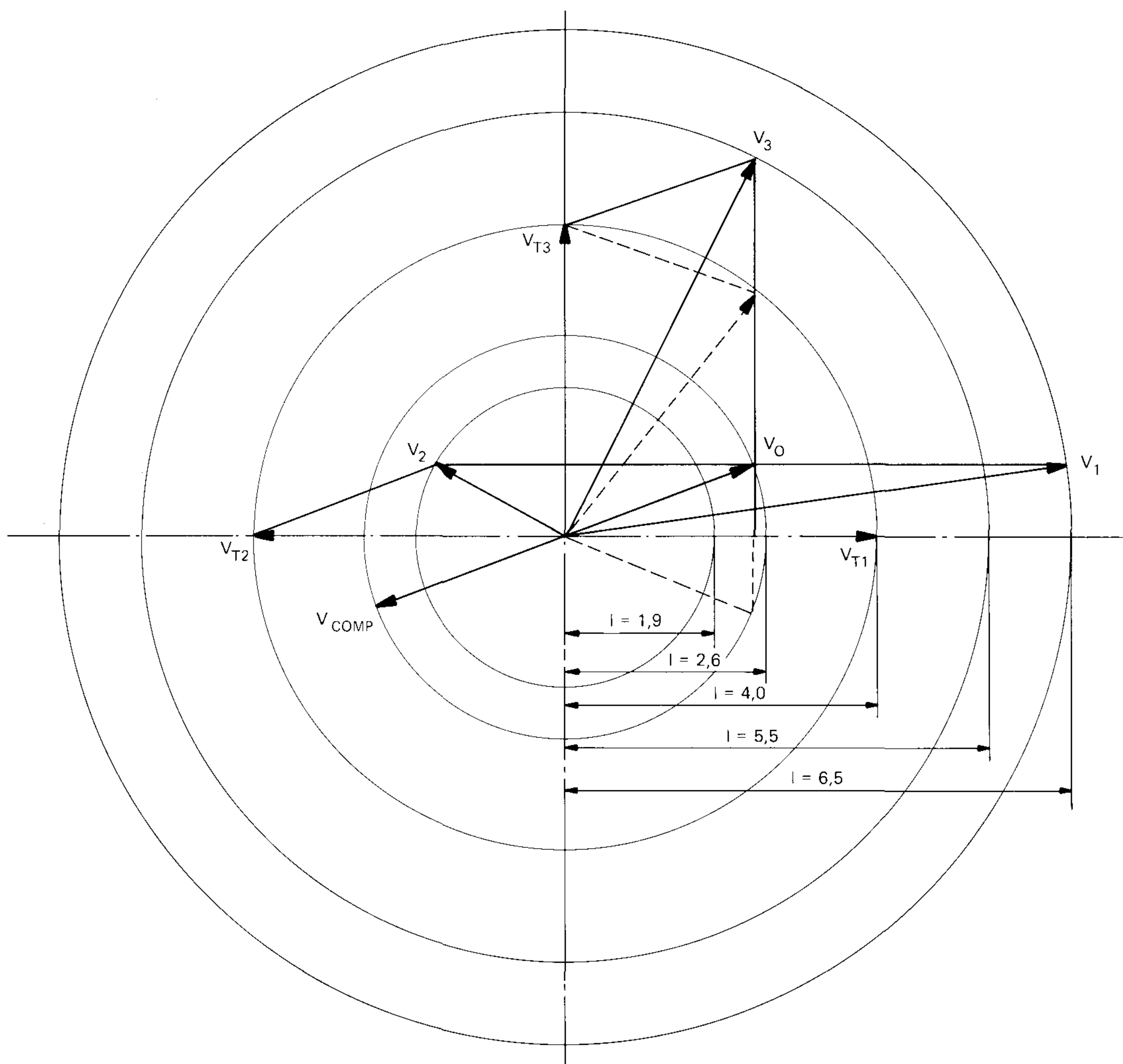
The machine was run to establish the vibration velocity level caused by the original unbalance, this was found to be  $V_0 = 2,6 \text{ mm/s}$ . A 10 g trial mass was fastened to the rotor in position 1, and the machine run again. This gave a vibration level of  $V_1 = 6,5 \text{ mm/s}$  due to the combined effect of the trial mass and the initial unbalanced mass. Before the next run, the trial mass was moved 180° round the rotor and fastened at the same radius in position 2. The machine was run, and the vibration level was found to have decreased to  $V_2 = 1,9 \text{ mm/s}$ .

It was now possible to start to draw the vector diagram, but only the vector lengths were known, not the angles. However, circles could be drawn about a common centre, each having a radius equivalent to a vector length, i.e. the measured vibration level. Referring to the Geometry diagram in Fig.17, two circles have been drawn with radii proportional to  $V_1$  and  $V_2$  the two resultants of the original unbalance and the trial mass fastened in two positions at 180° from each other. A radius has also been drawn at an arbitrary angle in each circle. Identical parallelograms have been constructed using each radius as a diagonal, and taking the line from the centre of the circles to the midpoint of the line joining the ends of  $V_1$  and  $V_2$  as a common side.

It will now be seen that if the angles of  $V_1$  and  $V_2$  can be arranged to produce a common side with length equivalent to  $V_0$ , then the diagram will give a true representation of the vectors, so that the other side in each parallelogram must be equivalent to  $V_T$ , the vibration caused by the trial mass alone in positions 1 and 2. Furthermore, the following relationships exist:

$$V_2^2 = V_{T2}^2 + V_0^2 - 2V_{T2} V_0 \cos \alpha$$





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Fig.18. Vector diagram for Example 3

$$V_1^2 = V_{T1}^2 + V_0^2 - 2V_{T1} V_0 \cos \beta$$

and as  $\cos \beta = -\cos \alpha$

The equation for  $V_1^2$  simplifies to,

$$V_1^2 = V_{T1}^2 + V_0^2 + 2V_{T1} V_0 \cos \alpha$$

$$\text{So that } V_{T1} = V_{T2}$$

$$= \sqrt{\frac{V_1^2 + V_2^2 - 2V_0^2}{2}}$$

$$\text{and } \alpha = \cos^{-1} \frac{V_1^2 - V_2^2}{4V_T V_0}$$

However, as  $\cos \alpha = \cos(-\alpha)$  it is not immediately obvious whether the vector for the original unbalance,  $V_0$  lies above or below the  $V_{T2} - V_{T1}$  axis (i.e. the line joining trial mass position 1 with position 2). Therefore it was necessary to make another test run, with the

trial mass fastened in position 3, to give Vector  $V_3$ . Strictly it was not necessary to draw this vector. If the vibration level was greater than that from the trial mass alone ( $V_T$ ),  $V_0$  would lie above the  $V_{T2} - V_{T1}$  axis (shown as full line in Fig.18). If the vibration level were less than  $V_T$ ,  $V_0$  would lie below the  $V_{T2} - V_{T1}$  axis, shown with broken line in Fig.18.

As a result of mounting the trial mass in position 3, when the machine was run, a vibration velocity level of  $V_3 = 5,5 \text{ mm/s}$  was recorded, thereby indicating that the original unbalance vector should lie above the  $V_{T2} - V_{T1}$  axis.

Substituting the vibration values of  $V_0$ ,  $V_1$ , and  $V_2$  into equation 4,

$$V_T = \sqrt{\frac{6,5^2 + 1,9^2 - 2 \times 2,6^2}{2}}$$

$$= 4 \text{ mm/s}$$

And now using the equation for  $\alpha$

$$\alpha = \cos^{-1} \frac{6,5^2 - 1,9^2}{4 \times 4 \times 2,6}$$

$$= \cos^{-1} 0,9288$$

$$= \pm 21,74^\circ$$

And because  $V_0$  has been found to lie above the  $V_{T2} - V_{T1}$  axis, i.e. at  $21,74^\circ$  from position 1 towards position 3, the compensation mass must be fastened at  $21,74^\circ$  below position 2.

The magnitude of the balancing mass is found as before,

$$M_{\text{COMP}} = M_0 = \frac{V_0}{V_T} \times M_T$$

$$= \frac{2,6 \times 10}{4}$$

$$= 6,5 \text{ g}$$



# Dynamic Balancing Measurement

## Example 4,

To make the measurements for balancing, both statically and dynamically, a machine that has a rigid rotor supported in two bearings; i.e. a balancing problem in two planes.

The measuring arrangement shown in Fig.10 was employed in this example, and the basic method used can be extended to solve balancing problems in more than two planes.

The procedure is the same as in the foregoing examples, finding the effect of a known trial mass attached to the rotor, except that now measurements have to be made in two planes at the two bearings, (designated planes 1 and 2). Two sets of measuring instruments can be used, with a set at each bearing, to determine the vibration levels and phase angles produced, so that all necessary data can be obtained from only three test runs. Alternatively, a single set of equipment can be used, with the accelerometer being moved from one bearing to the other, or the Vibration Meter can be switched between two accelerometers, one at each bearing.

To avoid special emphasis of high or low frequency components, vibration velocity was chosen as the measure of vibration level on the Vibration Meter, and the 3% bandwidth selected on the Tunable Band Pass Filter. The Phase Meter upper frequency limit was set at "2 kHz" to eliminate unwanted high frequency signals, while the lower limit was set at "2 Hz" to take advantage of the longer averaging time to obtain a steadier phase indication.

As a first step, the vibration velocity levels and phase angles had to be measured at each bearing to establish the magnitude of the original unbalance. The change over switch in the Trigger Unit Type 5767 was set to "Reference" so that the datum signal from the trigger source was divided, with one part passed directly to Input A of the Phase Meter, while the other part passed through the Band Pass Filter before reaching Input B. The

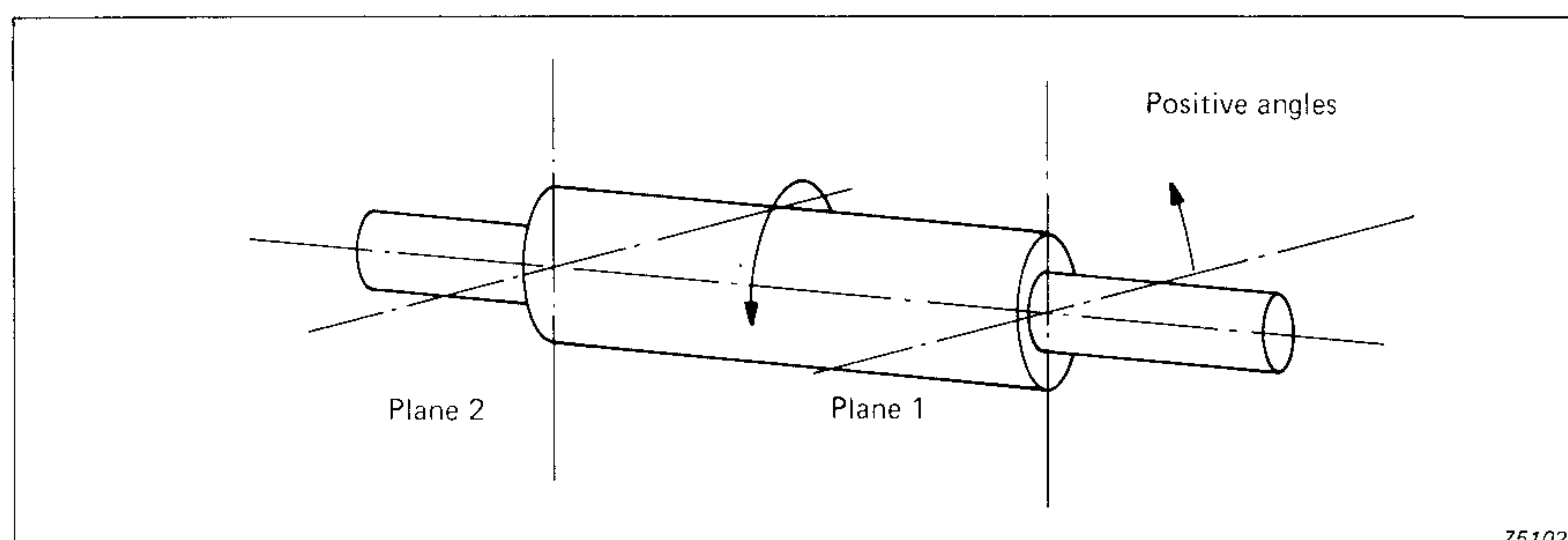


Fig.19. The rotor showing the measuring planes

slope switches on the Phase Meter were arranged with "+" on Channel A and "-" on Channel B to produce a stable reading of  $180^\circ$  for in-phase signals when measuring in the 0 to  $360^\circ$  range.

The machine was run while the Filter was tuned slowly through the frequency range where the rotational frequency was expected to lie. When the pointer of the analog meter stopped sweeping from one end of the scale to the other, and stabilized in the middle of the scale at  $\pi$ , the Filter was finely tuned with the aid of the digital display until a reading of  $180^\circ$  (3,14 rad) appeared. As the input and output of the Filter were now in phase, the centre frequency of the Filter had been accurately tuned to the rotational frequency of the machine, thereby eliminating any phase errors. The slope switches on the Phase Meter were set to the same sign, without changing the Filter setting. "Measure" was selected on the change over switch, so that the Filter operated on the vibration signal derived from the Accelerometer.

Vibration level (V) and phase angle ( $\gamma$ ) were measured at both bearings. Then a known trial mass was mounted at a known radius on the rotor bearing plane 1, where

balancing masses were to be fastened near one of the bearings. A test run was made to find the effect of the trial mass, both on bearing 1, and also on bearing 2. The machine was stopped and the trial mass moved to the other balancing plane, for convenience using the same radius and angular position as for the first plane. Another test run was made to find the effect on the bearings of the trial mass in its new position.

The results have been arranged in Table 1 and a vector notation for each measurement has also been included. The notation represents the complete vector, both vibration level (length) and phase angle, in one convenient term. It also indicates the plane in which the measurements were made, and the plane where the trial mass (if any) was fastened. So that  $V_{1,0}$  represents a vibration level V and a phase angle  $\gamma$  measured in Plane 1, with no trial mass on the rotor.  $V_{1,2}$  represents a vibration level V and phase angle  $\gamma$  measured in Plane 1 when a trial mass is fastened to Plane 2, and so on.

The measured vibration levels and angles from the Table could be used to draw vector diagrams similar to those shown in Fig.20.

Trial Mass Size and Location	Measured Effect of Trial Mass					
	Plane 1			Plane 2		
None	7,2 mm/s	238°	$\vec{V}_{1,0}$	13,5 mm/s	296°	$\vec{V}_{2,0}$
2,5 g on Plane 1	4,9 mm/s	114°	$\vec{V}_{1,1}$	9,2 mm/s	347°	$\vec{V}_{2,1}$
2,5 g on Plane 2	4,0 mm/s	79°	$\vec{V}_{1,2}$	12,0 mm/s	292°	$\vec{V}_{2,2}$

Table 1. Measured vibration levels and phase angles for Example 4



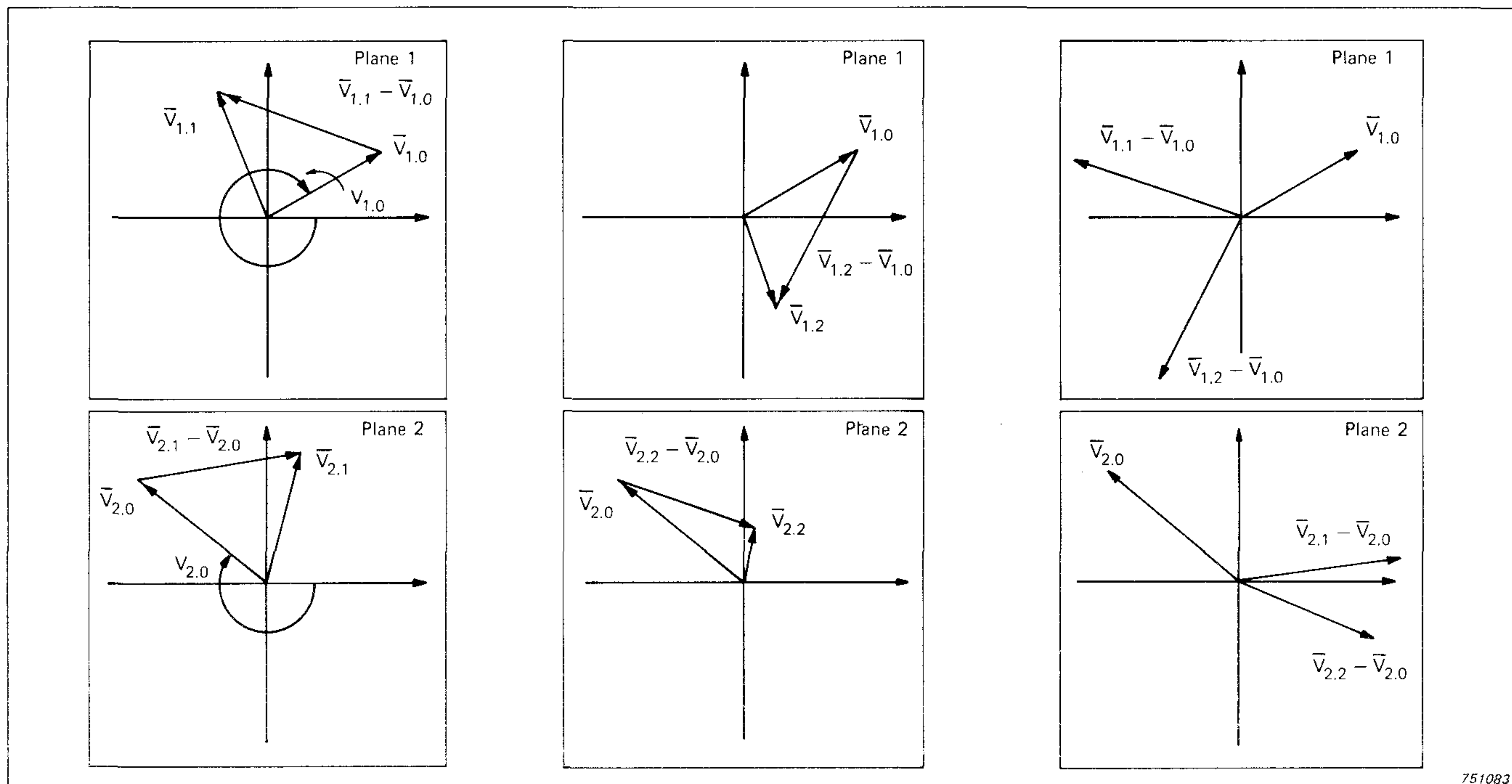


Fig.20. Vectorial representation of the vibration levels

## Dynamic Balancing Calculation

### Example 5,

To calculate the balancing masses and their positions from the data in Table 1, by means of an externally programmable calculator.

Using this method of calculation, an inexperienced operator can soon learn to perform the whole program in about two minutes.

The following procedure is for the Hewlett-Packard HP-67 and HP-97 calculators. The pre-recorded program is stored on a single magnetic card available from Brüel & Kjær on order number WW 9002. A method employing a Texas Instruments TI 59 calculator and program on a magnetic card (order number WW 9007) is presented in Appendix 1.

A mathematical basis for the solution of two plane balancing problems used by the calculator programs (and for the computer calculation later) is given in Appendix 2.

Always start with the calculator switched off to ensure freedom from all remnants of stored data from previous calculations.

1. Switch the calculator ON.
2. Read the program card into the calculator memory as described in the calculator instruction manual. (The program card must be passed through the reader twice since the program fills both tracks.)
3. Place the card in the slot above the user definable keys.
4. Key in the data in the following manner (i.e. in the order indicated in Table 1):

Velocity 1,0

ENTER ↑

Phase 1,0

f A

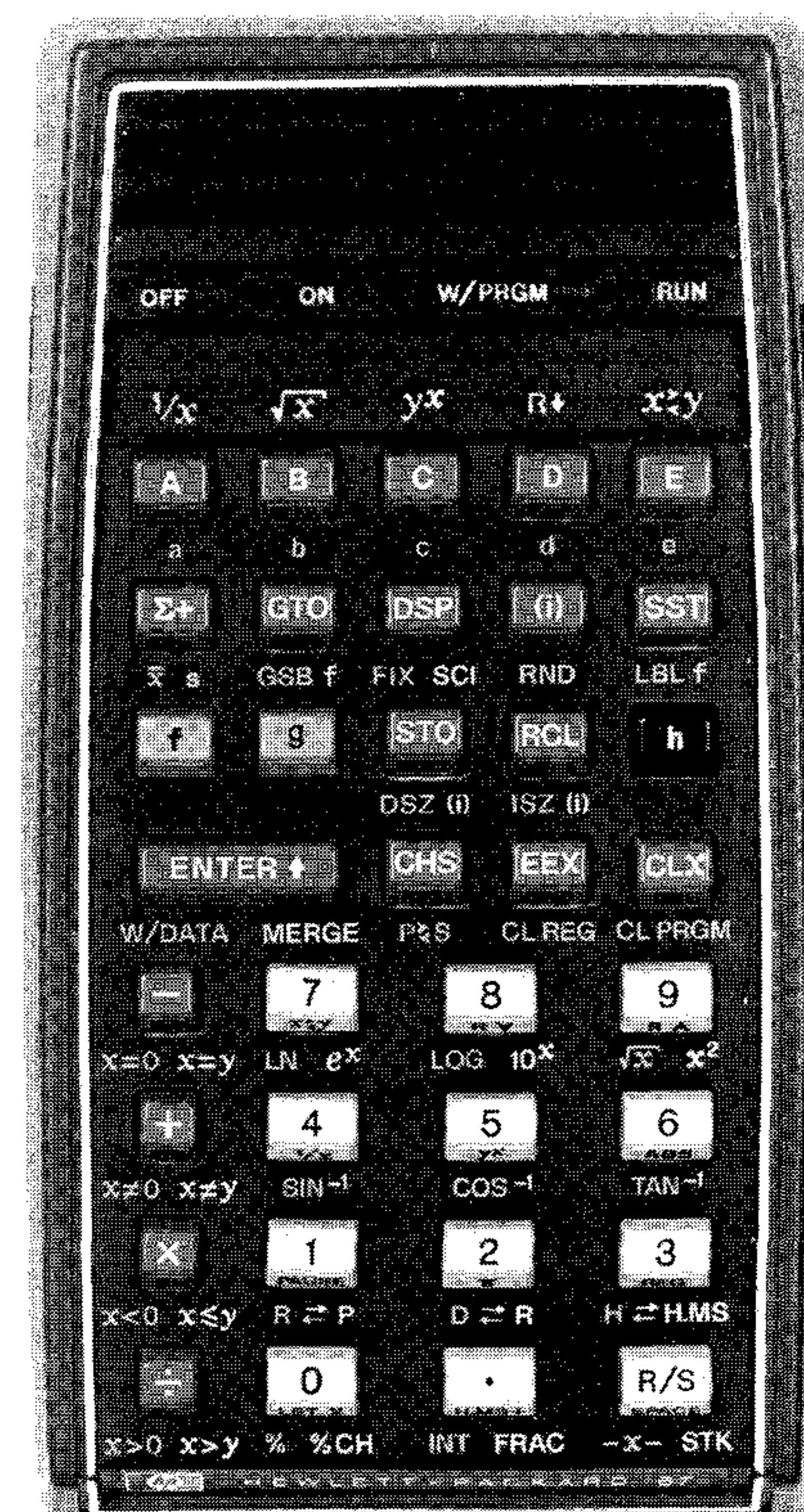


Fig.21. The Hewlett-Packard HP-67 Calculator



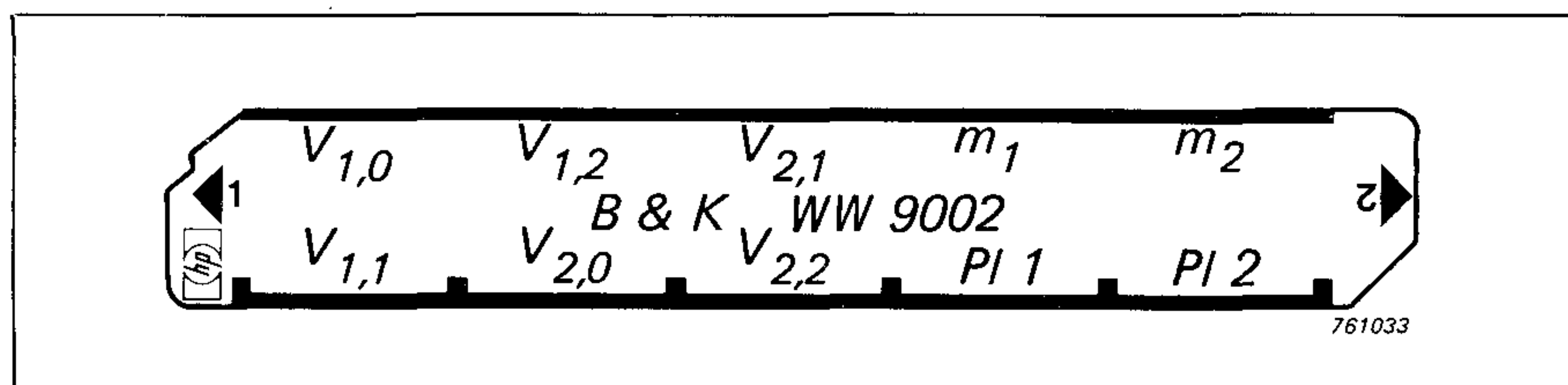


Fig.22. A program card for the HP-67

Wait a few seconds until the calculator gives a stable display. Then key in:

Velocity 1,1

**ENTER**

Phase 1,1

**A**

Key in the remaining values in the same manner using user-definable keys B and C according to the designations on the program card.

The values may be entered in any order, except for  $V_{2,2}$  which must be keyed in last. Before  $V_{2,2}$  is entered, the variables may be changed or corrected at will by simply keying in the new values. Once  $V_{2,2}$  has been entered however, the data entry routine is completed and if it is desired to change any of the input variables, all the values must be keyed in again.

5. Key in the trial mass ( $m_1$ ) for

plane 1 and press **f**  
**D**

6. Key in the trial mass ( $m_2$ ) for plane 2 and press **f**  
**E**

7. Press **D**. The balancing mass and angle for plane 1 are calculated. The calculation time is about 15 seconds. The HP-67 will first show the mass for about 5 seconds and then the angle. The HP-97 will print the two values.

8. Press **E**. The balancing mass and angle for plane 2 are calculated and displayed as for plane 1.

9. If it is desired to see the values again, press either **D** or **E**.

10. To perform the calculation on new data, enter data as described beginning with step 4. It is not necessary to clear the machine first.

## Example 6,

*Dynamic balancing using a computer to perform the calculations.*

For those with access to a computer, a program in BASIC is available (order number WW 1326 from B & K) to make the calculations. Fig.23 shows the program, plus the entry of data for a particular balancing problem, the calculated vector lengths and phase angles of the results.

Table 2 contains the vibration levels and phase angles recorded for a machine being balanced using a measuring arrangement similar to that shown in Fig.3.

The data for calculation must be entered into the computer in the following order,  $V_{1,0}$ ,  $\gamma_{1,0}$ ,  $V_{2,0}$ ,  $\gamma_{2,0}$ ,  $V_{1,1}$ ,  $\gamma_{1,1}$ ,  $V_{2,1}$ ,  $\gamma_{2,1}$ ,  $V_{1,2}$ ,  $\gamma_{1,2}$ ,  $V_{2,2}$ , and  $\gamma_{2,2}$ . The vector lengths and phase angles for each plane are shown entered at statement 499 in the program. Total elapsed time to enter the data and make the calculation on a B & K/Varian Computer Type 7504 was of the order of two minutes.

The compensation masses to balance the rotor were calculated as in the previous example:

Plane 1  $M_{\text{COMP}}$

$$= 1,721 \times 1,15 \text{ g} \\ = 1,98 \text{ g at } 236,2^\circ$$

Plane 2  $M_{\text{COMP}}$

$$= 0,931 \times 1,15 \text{ g} \\ = 1,07 \text{ g at } 121,8^\circ$$

When masses with these values had been fastened at the correct angles and radius on the test rotor, new vibration levels were measured:

Plane 1 vibration level = 22 mm/s, an improvement of 87%,

Plane 2 vibration level = 8,5 mm/s, an improvement of 84%.

Trial Mass Size and Location	Measured Effect of Trial Mass					
	Plane 1			Plane 2		
None	170 mm/s	112°	$\vec{V}_{1,0}$	53 mm/s	78°	$\vec{V}_{2,0}$
1,15 g on Plane 1	235 mm/s	94°	$\vec{V}_{1,1}$	58 mm/s	68°	$\vec{V}_{2,1}$
1,15 g on Plane 2	185 mm/s	115°	$\vec{V}_{1,2}$	77 mm/s	104°	$\vec{V}_{2,2}$

Table 2. Measured vibration levels and phase angles for Example 6



DYNBAL WW 1326

LIST

```

10 DIM C(2,2),D(2,2),E(2,2),F(2,2),G(2,2),H(2,2),I(2,2),J(2,2)
12 DIM K(2,2),L(2,2),M(2,2),N(2,2),O(2,2),P(2,2),Q(2,2),R(2,2)
14 DIM S(2,2),T(2,2),U(2,2),V(2,2),X(2,2)
20 FOR Y= 1 TO 6
30 READ A(Y),B(Y)
35 LET B(Y)=B(Y)*ATN( 1)/ 45
40 NEXT Y
50 LET C( 1, 1)=A( 1)*COS(B( 1))
60 LET C( 1, 2)=A( 1)*SIN(B( 1))
62 LET C( 2, 1)=-C( 1, 2)
65 LET C( 2, 2)=C( 1, 1)
70 LET D( 1, 1)=A( 2)*COS(B( 2))
75 LET D( 1, 2)=A( 2)*SIN(B( 2))
80 LET D( 2, 1)=-D( 1, 2)
85 LET D( 2, 2)=D( 1, 1)
90 LET E( 1, 1)=A( 3)*COS(B( 3))
95 LET E( 1, 2)=A( 3)*SIN(B( 3))
100 LET E( 2, 1)=-E( 1, 2)
105 LET E( 2, 2)=E( 1, 1)
110 LET F( 1, 1)=A( 4)*COS(B( 4))
115 LET F( 1, 2)=A( 4)*SIN(B( 4))
120 LET F( 2, 1)=-F( 1, 2)
125 LET F( 2, 2)=F( 1, 1)
130 LET G( 1, 1)=A( 5)*COS(B( 5))
135 LET G( 1, 2)=A( 5)*SIN(B( 5))
140 LET G( 2, 1)=-G( 1, 2)
145 LET G( 2, 2)=G( 1, 1)
150 LET H( 1, 1)=A( 6)*COS(B( 6))
155 LET H( 1, 2)=A( 6)*SIN(B( 6))
160 LET H( 2, 1)=-H( 1, 2)
165 LET H( 2, 2)=H( 1, 1)
200 MAT I=E-C
205 MAT J=F-D
210 MAT K=G-C
215 MAT L=F-D
220 MAT M=H-D
225 MAT N=E-C
230 MAT O=D*I
235 MAT P=C*J
240 MAT Q=K*L
245 MAT R=M*N
250 MAT S=O-P
255 MAT T=Q-R
260 MAT U=INV(T)
265 MAT V=S*U
270 MAT I=C*M
275 MAT J=D*K
280 MAT K=I-J
285 MAT X=K*U
290 LET Y1=SQR(V( 1, 1)+ 2+V( 1, 2)+ 2)
300 LET Y2=SQR(X( 1, 1)+ 2+X( 1, 2)+ 2)
310 IF V( 1, 1)< 0 THEN 340
320 LET Y3= 0
330 GOTO 350
340 LET Y3= 180
350 IF X( 2, 2)< 0 THEN 380
360 LET Y4= 0
370 GOTO 390
380 LET Y4= 180
390 LET Y5=Y3+(ATN(V( 1, 2)/V( 1, 1)))/ATN( 1)* 45
400 LET Y6=Y4+(ATN(X( 1, 2)/X( 1, 1)))/ATN( 1)* 45
410 PRINT "MØDULUS AND ARGUMENT ØF Q1:",Y2,Y6
420 PRINT "MØDULUS AND ARGUMENT ØF Q2:",Y1,Y5
499 DATA 170, 112, 53, 78, 235, 94, 58, 68, 185, 115, 77, 104
510 END
RUN
MØDULUS AND ARGUMENT ØF Q1: 1.72127 236.17
MØDULUS AND ARGUMENT ØF Q2: .930879 121.844
READY

```

Fig.23. Dynamic balancing program in BASIC

## Example 7,

*Calculation of the balancing masses and their positions purely with the aid of a non-programmable mathematical pocket calculator, time taken approximately one hour with practice.*

In theory pure arithmetic, or a slide rule could also have been used, but calculation times would have increased greatly. Working with results of Example 4 which were drawn as vector diagrams in Fig.20, it will be seen that in terms of vector notation:

$V_{1,1} - V_{1,0}$  is the effect in Plane 1 of a trial mass in Plane 1,

$V_{1,2} - V_{1,0}$  is the effect in Plane 1 of a trial mass in Plane 2,

$V_{2,1} - V_{2,0}$  is the effect in Plane 2 of a trial mass in Plane 1,

$C_{2,2} - V_{2,0}$  is the effect in Plane 2 of a trial mass in Plane 2.

Each of the vector diagrams in Fig.20 is analogous to the vector diagrams in the earlier examples. However, as described more fully in Appendix 2, when an unbalanced mass (trial mass) is applied at one measuring plane, it has an effect on both planes. Therefore for complete balance of the rotor, masses must be added at both balancing planes in such a way that produces vibration vectors with equal magnitude (length to  $V_{1,0}$  and  $V_{2,0}$ , but which have opposite phase angles. Mathematically the problem is to find two vector operators  $\hat{Q}_1$  (with vector length  $Q_1$  and phase angle  $\gamma_1$ ) and  $\hat{Q}_2$  ( $Q_2$  and  $\gamma_2$ ) which satisfy the following equations.

$$Q_1 (V_{1,1} - V_{1,0}) + Q_2 (V_{1,2} - V_{1,0}) = -V_{1,0} \quad \text{Eqn.6}$$

$$Q_1 (V_{2,1} - V_{2,0}) + Q_2 (V_{2,2} - V_{2,0}) = -V_{2,0} \quad \text{Eqn.7}$$

The vectors in these expressions can be resolved conveniently by means of complex number arithmetic, which allows equations 6 and 7 to be solved as a pair of equations with 2 unknowns  $Q_1$  and  $Q_2$ . First find  $Q_1$  in terms of  $Q_2$ ,

$$Q_1 = \frac{-V_{1,0} - Q_2 (V_{1,2} - V_{1,0})}{V_{1,1} - V_{1,0}} \quad \text{Eqn.8}$$

and then solving for  $Q_2$ ,

$$Q_2 = \frac{V_{2,0}(V_{1,1} - V_{1,0}) - V_{1,0}(V_{2,1} - V_{2,0})}{(V_{2,1} - V_{2,0})(V_{1,2} - V_{1,0}) - (V_{2,2} - V_{2,0})(V_{1,1} - V_{1,0})} \quad \text{Eqn.9}$$

The measured values of vibration level and phase angle in Table 1 are the polar coordinates for the vector quantity  $V$ . When a Cartesian system of coordinates is used, with real and imaginary components where

$$\vec{V} = a + jb \quad \text{Eqn.10}$$

a mathematical solution for Equations 8 and 9 can be calculated.

Polar coordinates can be converted to Cartesian by use of the two equations:

$$a = V \cos \gamma \quad \text{Eqn.11}$$

$$b = V \sin \gamma \quad \text{Eqn.12}$$

Then the values in Table 3 can be calculated by again applying the rules of complex arithmetic (see Table 4), for example:

$$\begin{aligned} V_{1,1} - V_{1,0} &= (-2,0 + 4,48j) - (-3,82 - 6,12j) \\ &= (+1,83 + 10,6j) \end{aligned}$$

Substituting real and imaginary values into Equation 9:

$$Q_2 = \frac{(+5,92 - 12,13j)(+1,82 + 10,60j) - (-3,82 - 6,12j)(+3,04 + 10,06j)}{(+3,04 + 10,06j)(+4,58 + 10,05j) - (-1,42 + 1,00j)(+1,82 + 10,60j)}$$

Which simplifies (again by means of complex number arithmetic) to:

$$Q_2 = +0,1598 - 1,1264j$$

Which can be reconverted to polar coordinates by means of the following equations:

$$V = +\sqrt{a^2 + b^2} \quad \text{Eqn.13}$$

$$\text{for } a > 0 \quad \gamma = \tan^{-1} \frac{b}{a} \quad -90^\circ < \gamma < +90^\circ \quad \text{Eqn.14}$$

$$\text{for } a < 0 \quad \gamma = 180^\circ + \tan^{-1} \frac{b}{a} \quad +90^\circ < \gamma < +270^\circ \quad \text{Eqn.15}$$

$$\text{So that vector length, } Q_2 = 1,1376$$

$$\text{and phase angle, } \gamma_2 = -81,9^\circ$$

Now these values can be substituted into Equation 8 so that  $Q_1$  can be found.

$$Q_1 = \frac{-(-3,82 - 6,12j) - (+4,48 + 10,05j)(+0,1598 - 1,1264j)}{(+1,82 + 10,6j)}$$

Addition:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

Subtraction:

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

Multiplication:

$$(a + jb)(c + jd) = (ac - bd) + j(bc + ad)$$

Division:

$$\frac{a + jb}{c + jd} = \frac{ac + bd}{c^2 + d^2} + j \frac{bc - ad}{c^2 + d^2}$$

Table 4. Rules for complex number arithmetic



Which simplifies to:

$$Q_1 = + 0,7468 + 0,9033j$$

And using Equation 13 and 14, the vector length and phase angle are found:

$$Q_1 = 1,1720$$

$$q_1 = + 50,4^\circ$$

Therefore the balancing masses to counteract the original unbalance of the rotor are as follows:

Plane 1  $M_{COMP}$

$$= 1,172 \times 2,5 \text{ g}$$

$$= 2,93 \text{ g at } + 50,4^\circ$$

( $50,4^\circ$  from the trial mass position in the direction of rotation).

Plane 2  $M_{COMP}$

$$= 1,1376 \times 2,5 \text{ g}$$

$$= 2,84 \text{ g at } -81,9^\circ$$

( $81,9^\circ$  from the trial mass position in the opposite direction from the rotation).

Masses with these values were fastened in the respective planes on the rotor at the calculated angles, and at the radius used previously for the trial masses. A test run was made to assess the quality of the balance. Its results were as follows:

Plane 1 vibration level =  $0,5 \text{ mm/s}$ , which represents a reduction in vibration velocity level of 93% from the original  $7,2 \text{ mm/s}$ .

Plane 2 vibration level =  $0,4 \text{ mm/s}$ , which represents a reduction in vibration velocity level of 97% from the original  $13,5 \text{ mm/s}$ .

As an added test, the two balance masses were moved through an angle of  $10^\circ$  to find the importance of the phase angle determination. When the machine was run again, the vibration velocity level at Plane 1 was found to be  $1,8 \text{ mm/s}$ , with  $2,2 \text{ mm/s}$  at Plane 2. These results illustrate the value of the really accurate phase angle determination possible with the Type 2971 Phase Meter.

V	V	$\gamma$	a	jb
$\vec{V}_{1,0}$	7,2	$238^\circ$	-3,82	-6,12j
$\vec{V}_{1,1}$	4,9	$114^\circ$	-2,0	+ 4,48j
$\vec{V}_{1,2}$	4,0	$79^\circ$	+ 0,76	+ 3,93j
$\vec{V}_{2,0}$	13,5	$296^\circ$	+ 5,92	-12,13j
$\vec{V}_{2,1}$	9,2	$347^\circ$	+ 8,96	-2,07j
$\vec{V}_{2,2}$	12,0	$292^\circ$	+ 4,5	-11,13j
$(\vec{V}_{1,1} - \vec{V}_{1,0})$			+ 1,82	+ 1060j
$(\vec{V}_{2,1} - \vec{V}_{2,0})$			+ 3,04	+ 10,06j
$(\vec{V}_{1,2} - \vec{V}_{1,0})$			+ 4,58	+ 10,05j
$(\vec{V}_{2,2} - \vec{V}_{2,0})$			-1,42	+ 1,00j

Table 3. Conversion of coordinates in Example 7

## Appendix 1,

*Calculation of balancing masses and their positions by means of a Texas Instruments TI 59.*

This is a calculator that can be externally programmed by magnetic cards. The balancing program is contained on a single memory card that may be obtained from B & K on order number WW 9007. The calculation procedure is as follows:

1. Switch the calculator ON.
2. Read the program card into the calculator as described in its instruction manual. Be sure to press **CLR** between reading the two memory tracks.
3. Place the card in the slot above the user defined keys.
4. Press **2nd** **E** if the TI 59 is used in conjunction with Printer PC 100 A, and the results will be printed out directly.
5. Enter data in any order, using the following procedure:  
for example Velocity 2,2 press **X $\geq$ t**  
and Phase 2,2 press the user defined key corresponding to

the identity on the program card, i.e. Velocity **2,2**  
**X $\geq$ t** Phase 2,2 **2nd**  
**C**.

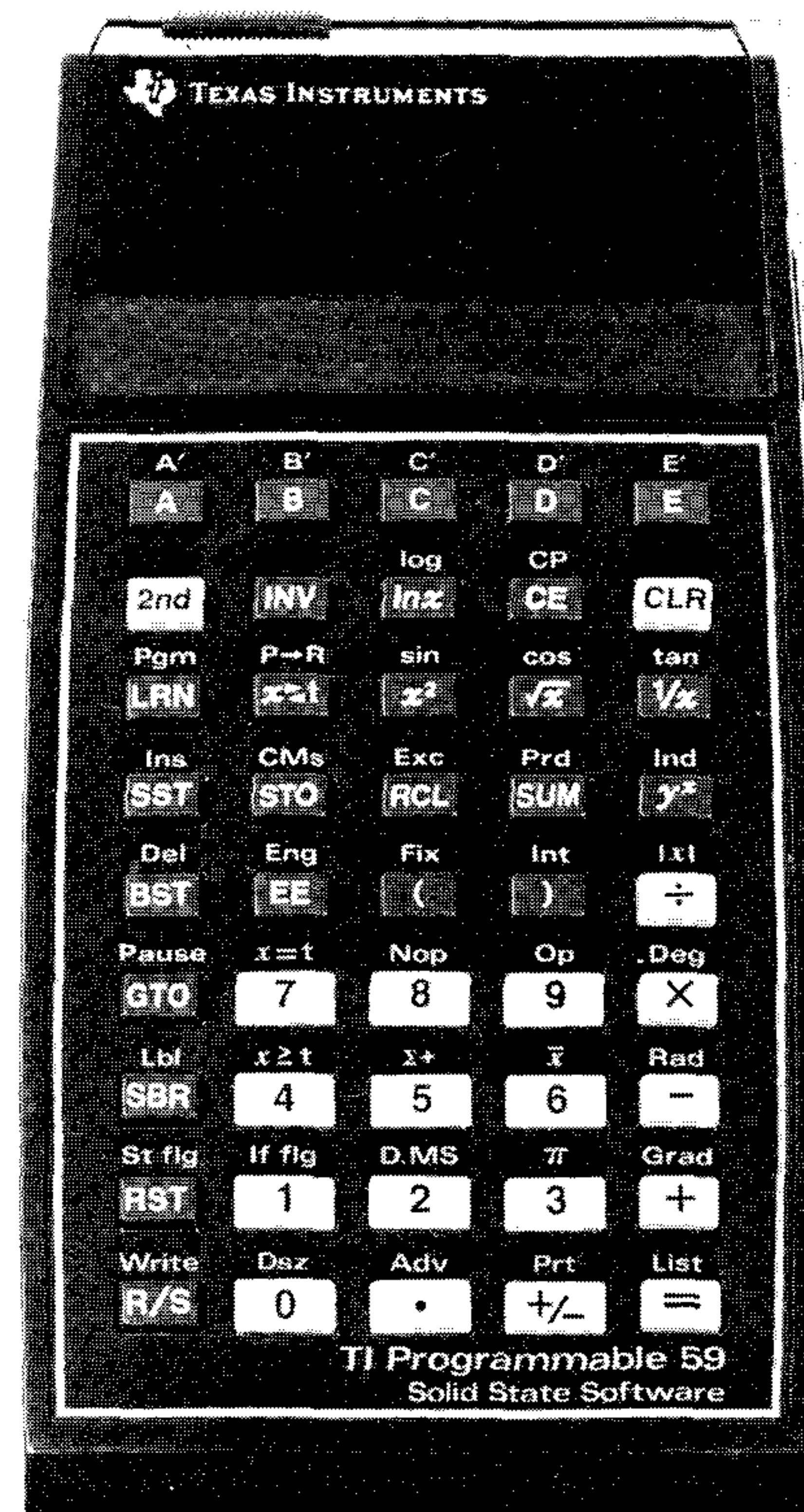


Fig.24. The Texas Instruments TI 59 Calculator

<div style="display: flex; justify-content: space-between; align-items: center;"> <span>◀</span> <span>▶</span> </div> <div style="text-align: center; margin-top: 5px;"> <b>TEXAS INSTRUMENTS</b>  <b>DYNBAL WW 9007</b> </div>				
V 1.1	V 2.0	V 2.2	M 1.2	PRINT
V 1.0	V 1.2	V 2.1	1.V2	2.V2

Fig.25. Program cards for the TI 59

6. Key in the trial mass for Plane 1, and press **[X↔t]**.
7. Key in the trial mass for Plane 2, and press **[2nd]** **[D]**.
8. Press **[D]** and the balancing mass and angle for Plane 2 are calculated. The mass appears on the display. Press **[X↔t]** to obtain the phase on the display. Calculation time is approximately 1 1/2 minutes.
9. Press **[E]** to obtain the Plane 1 data.
10. Press **[2nd]** **[E]** if the Printer is being used with the TI 59 to obtain a printout of data from both Planes without restarting.
11. Repeat steps 5 to 9 to perform a new balancing calculation.

12. If some input data has to be corrected during the read-in procedure, only step 5 has to be performed again for the actual velocity and phase.

## Appendix 2,

*Mathematical analysis of the problem of two plane balancing, using complex numbers to achieve a solution.*

Each of the vector diagrams in Fig.20 (reproduced again in Fig.26 for convenience) is analogous to the vector diagrams illustrating earlier examples. However, in the two plane balancing case, when an unbalanced (trial) mass is applied in

one measuring plane, it has an effect on both measuring planes. Therefore to balance the rotor completely, masses must be added in both balancing planes in such a way that their combined vibration vectors in each plane cancel out the original vibration vectors  $V_{1,0}$  and  $V_{2,0}$ . This can be achieved by changing the mass and angular position of the trial masses in such a way as to change the magnitudes and directions of the difference vectors:  $V_{1,1} - V_{1,0}$ ,  $V_{1,2} - V_{1,0}$ , etc.

In order to describe these changes, a vector operator  $\hat{Q}$  will be used, characterized by an amplitude  $Q$  and an angle  $\gamma$ . When  $\hat{Q}$  is operating on a vector  $\bar{V}$ , the resulting vector  $\hat{Q}(\bar{V})$  is derived from  $\bar{V}$  by multiplying the magnitude of  $\bar{V}$  by  $Q$ , and changing the direction by the angle  $\gamma$ . For example the operator  $\hat{Q}(Q=2, \gamma=45^\circ)$  will have the effect shown in Fig.27. If  $\bar{V}$  is the vibration level due to the trial mass,  $\hat{Q}(\bar{V})$  is the vibration level caused by a mass twice that of the trial mass applied at an angle  $45^\circ$  in the direction of rotation from the trial mass.

Letting the operators  $\hat{Q}_1$  and  $\hat{Q}_2$  describe the changes in the two measuring planes relative to the applied trial masses, the balancing problem can be expressed mathematically as:

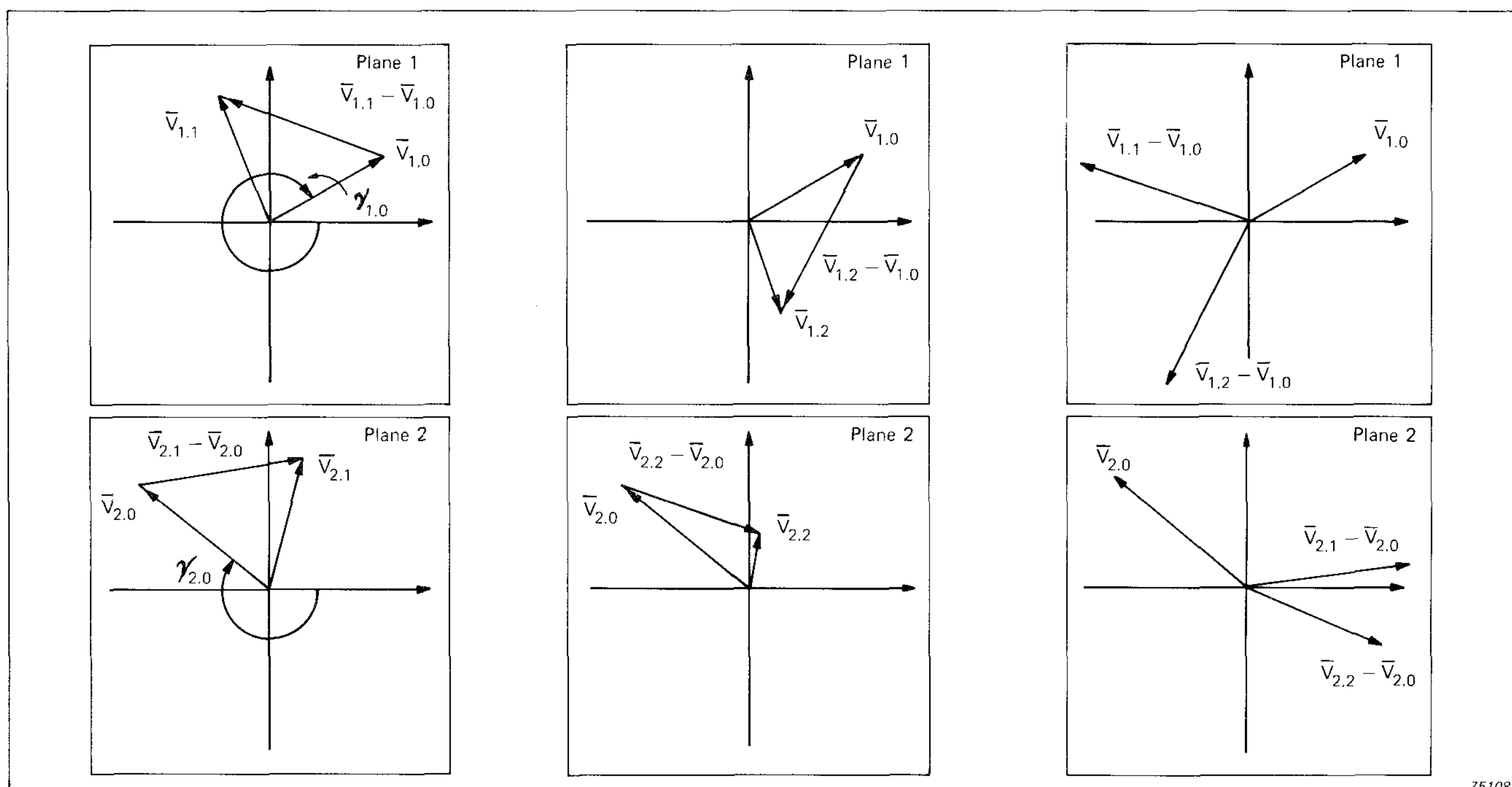


Fig.26. Vectorial representation of the vibration levels



$$\hat{Q}_1 (\bar{V}_{1,1} - \bar{V}_{1,0}) + \hat{Q}_2 (\bar{V}_{1,2} - \bar{V}_{1,0}) = -\bar{V}_{1,0} \quad \text{Eqn.6}$$

$$\hat{Q}_1 (\bar{V}_{2,1} - \bar{V}_{2,0}) + \hat{Q}_2 (\bar{V}_{2,2} - \bar{V}_{2,0}) = -\bar{V}_{2,0} \quad \text{Eqn.7}$$

The most convenient method of solving these equations is by the use of complex arithmetic. All of the vectors  $V$  are two dimensional and situated in the plane perpendicular to the axis of rotation, which can be treated as a complex plane, therefore they can be expressed as complex numbers. For example, the vector  $V$  lying in the complex plane in Fig.28 can be described as complex number  $V$  in two alternative ways. With Cartesian coordinate notation the real and imaginary components form the complex number:

$$\bar{V} = a + jb \quad \text{Eqn.10}$$

while with polar coordinate notation, modulus  $r$  and argument  $\gamma$  allow  $V$  to be written as:

$$\bar{V} = re^{j\gamma} = r(\cos \gamma + j \sin \gamma)$$

Transformations between the two special notations are governed by the following equations:

$$a = r \cos \gamma$$

$$b = r \sin \gamma$$

$$r = \sqrt{a^2 + b^2}$$

$$\gamma = \tan^{-1} \frac{b}{a} \quad a > 0$$

$$\gamma = 180^\circ + \tan^{-1} \frac{b}{a} \quad a < 0$$

Consider two complex numbers:

$$V_1 = a + jb = r_1 e^{j\gamma_1}$$

$$V_2 = c + jd = r_2 e^{j\gamma_2}$$

Using the most convenient coordinate representation, the arithmetical rules can be written as follows:

$$V_1 + V_2 = (a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$V_1 - V_2 = (a + jb) - (c + jd) = (a - c) + j(b - d)$$

$$V_1 \times V_2 = r_1 e^{j\gamma_1} \times r_2 e^{j\gamma_2} = (r_1 r_2) e^{j(\gamma_1 + \gamma_2)}$$

$$V_1 \div V_2 = \frac{r_1 e^{j\gamma_1}}{r_2 e^{j\gamma_2}} = \frac{r_1}{r_2} e^{j(\gamma_1 - \gamma_2)}$$

The method of multiplication is particularly useful when  $V_1$  is considered as the operator acting on  $V_2$ . The effect is that the amplitude of  $V_2$  is multiplied by  $V_1$ , and to change the direction of  $V_2$  by angle  $\gamma_1$  as shown in Fig.29. It will be recognized that the effect of  $V_1$  on  $V_2$  is exactly the same as that of the operator  $\hat{Q}$ , which means that in the complex plane the operator  $\hat{Q}(Q\gamma)$  is simply an ordinary multiplication with the complex number  $Qe^{j\gamma}$ .

Rewriting equations 6 and 7 with all  $Q$  and  $V$  terms expressed as complex numbers:

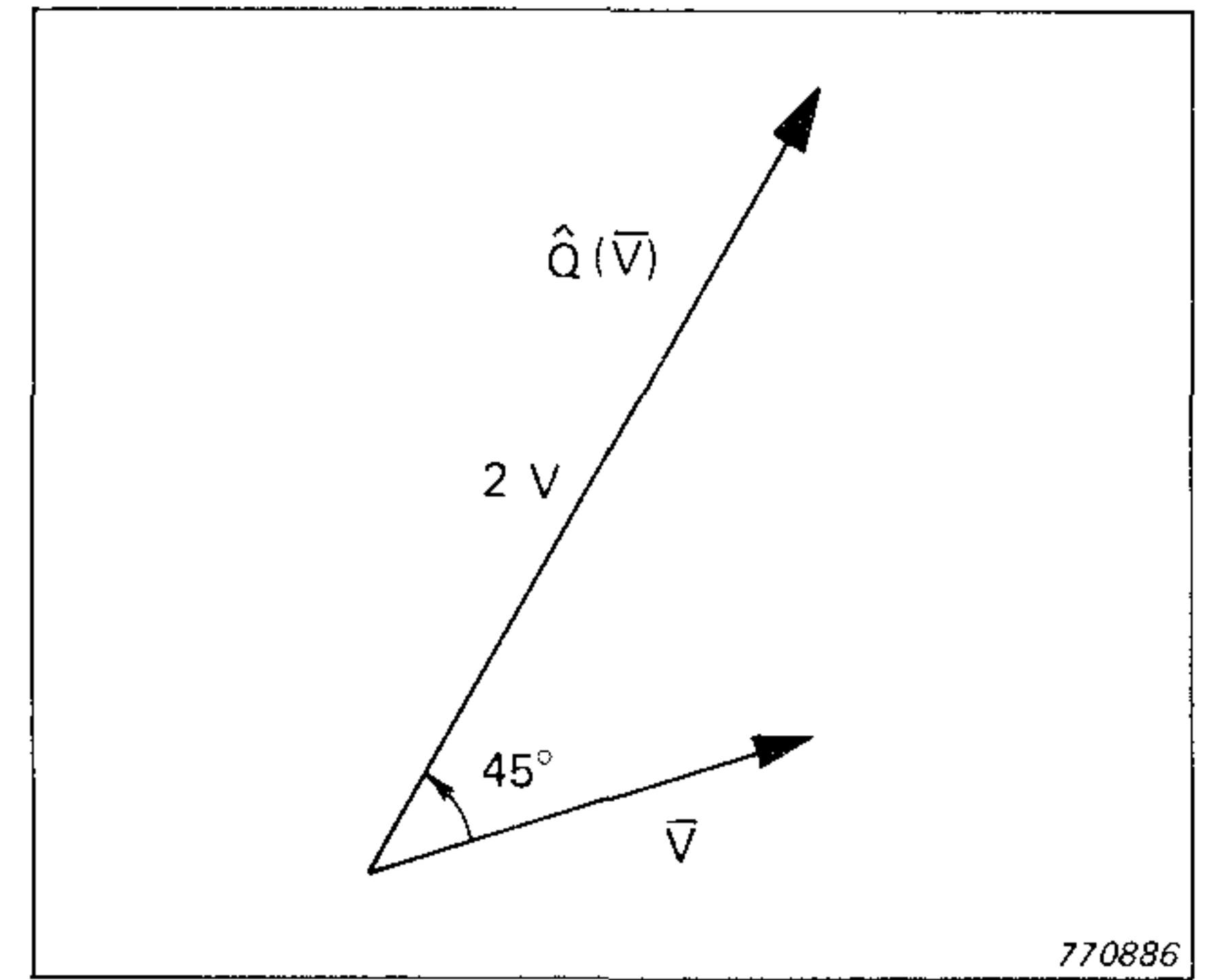


Fig.27. The effect of operator  $Q$  on vector  $V$

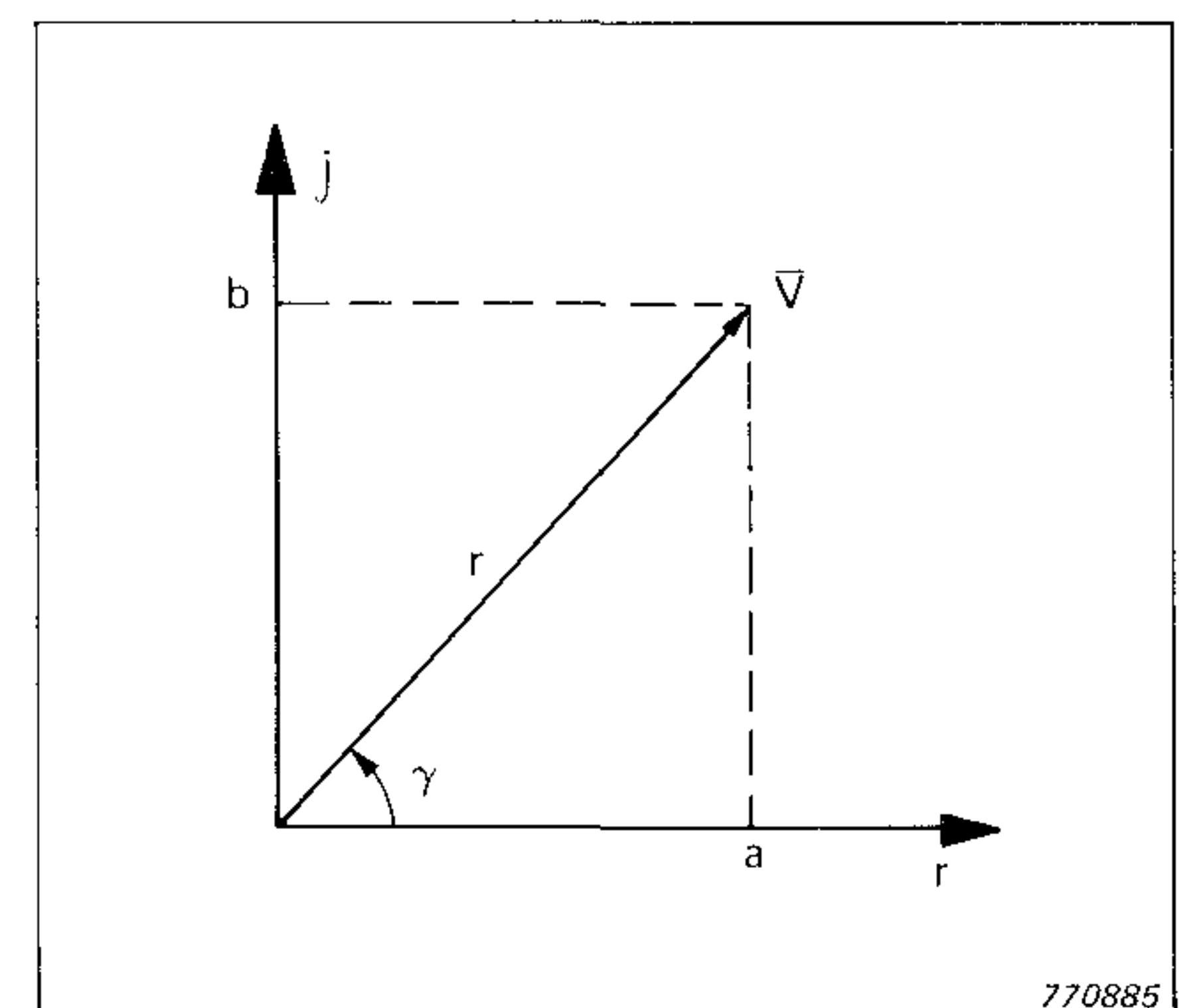


Fig.28. Real and complex planes

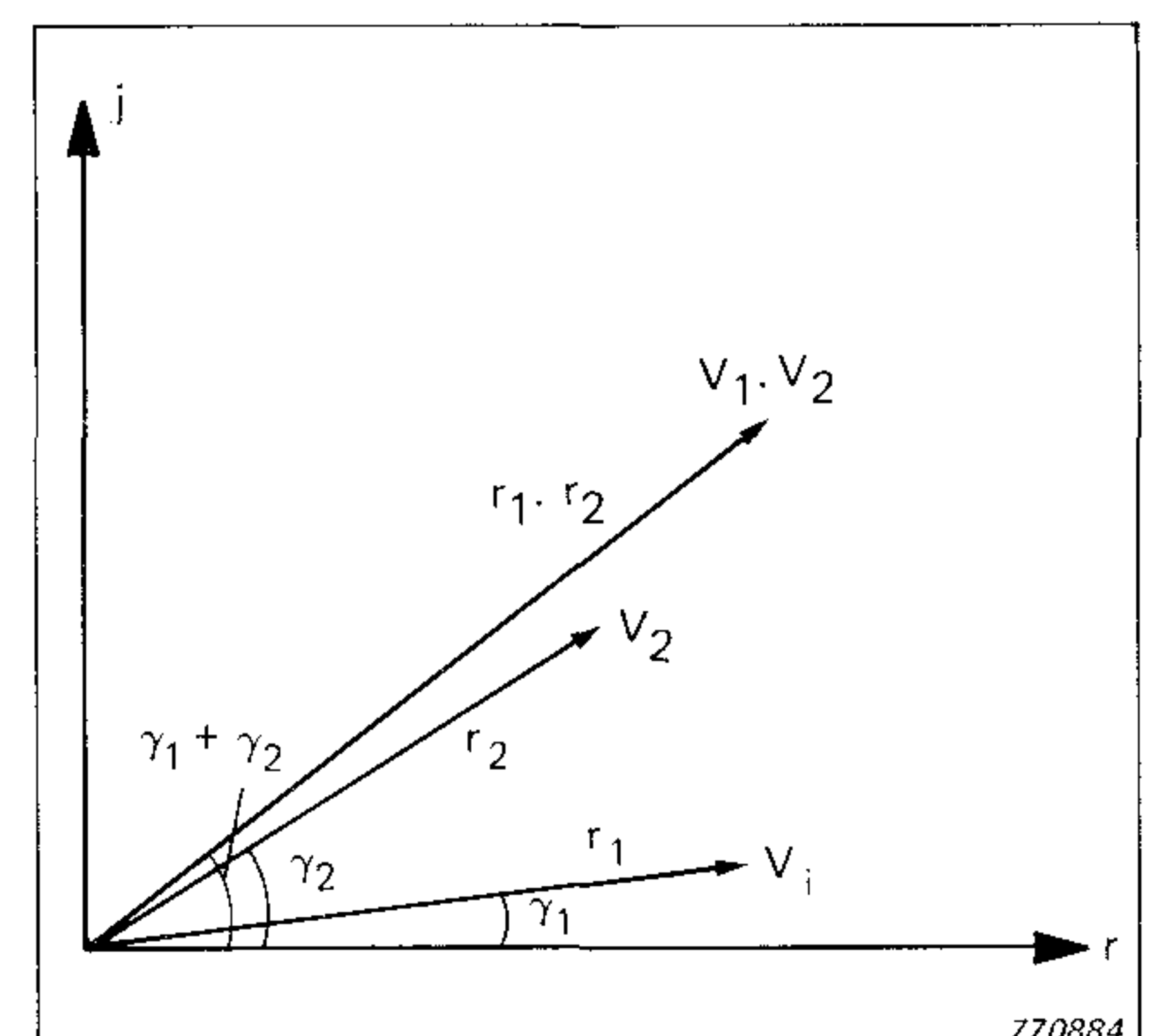


Fig.29. Vector operators



$$\hat{Q}_1 (\bar{V}_{1,1} - \bar{V}_{1,0}) + \hat{Q}_2 (\bar{V}_{1,2} - \bar{V}_{1,0}) = -\bar{V}_{1,0} \quad \text{Eqn.6}$$

$$\hat{Q}_1 (\bar{V}_{2,1} - \bar{V}_{2,0}) + \hat{Q}_2 (\bar{V}_{2,2} - \bar{V}_{2,0}) = -\bar{V}_{2,0} \quad \text{Eqn.7}$$

Solving equation 6 for  $Q_1$ :

$$Q_1 = \frac{-V_{1,0} - Q_2 (V_{1,2} - V_{1,0})}{(V_{1,1} - V_{1,0})} \quad \text{Eqn.8}$$

substituting  $Q_1$  in equation 7,

$$Q_2 = \frac{V_{2,0}(V_{1,1} - V_{1,0}) - V_{1,0}(V_{2,1} - V_{2,0})}{(V_{2,1} - V_{2,0})(V_{1,2} - V_{1,0}) - (V_{2,2} - V_{2,0})(V_{1,1} - V_{1,0})} \quad \text{Eqn.9}$$

The equations for  $Q_1$  and  $Q_2$  can be solved by inserting the complex values for the vector quantities, as is shown in Example 7. This produces real and imaginary values for the vector operators that can be converted to a vector length and phase angle with the aid of equations 13, 14 and 15.