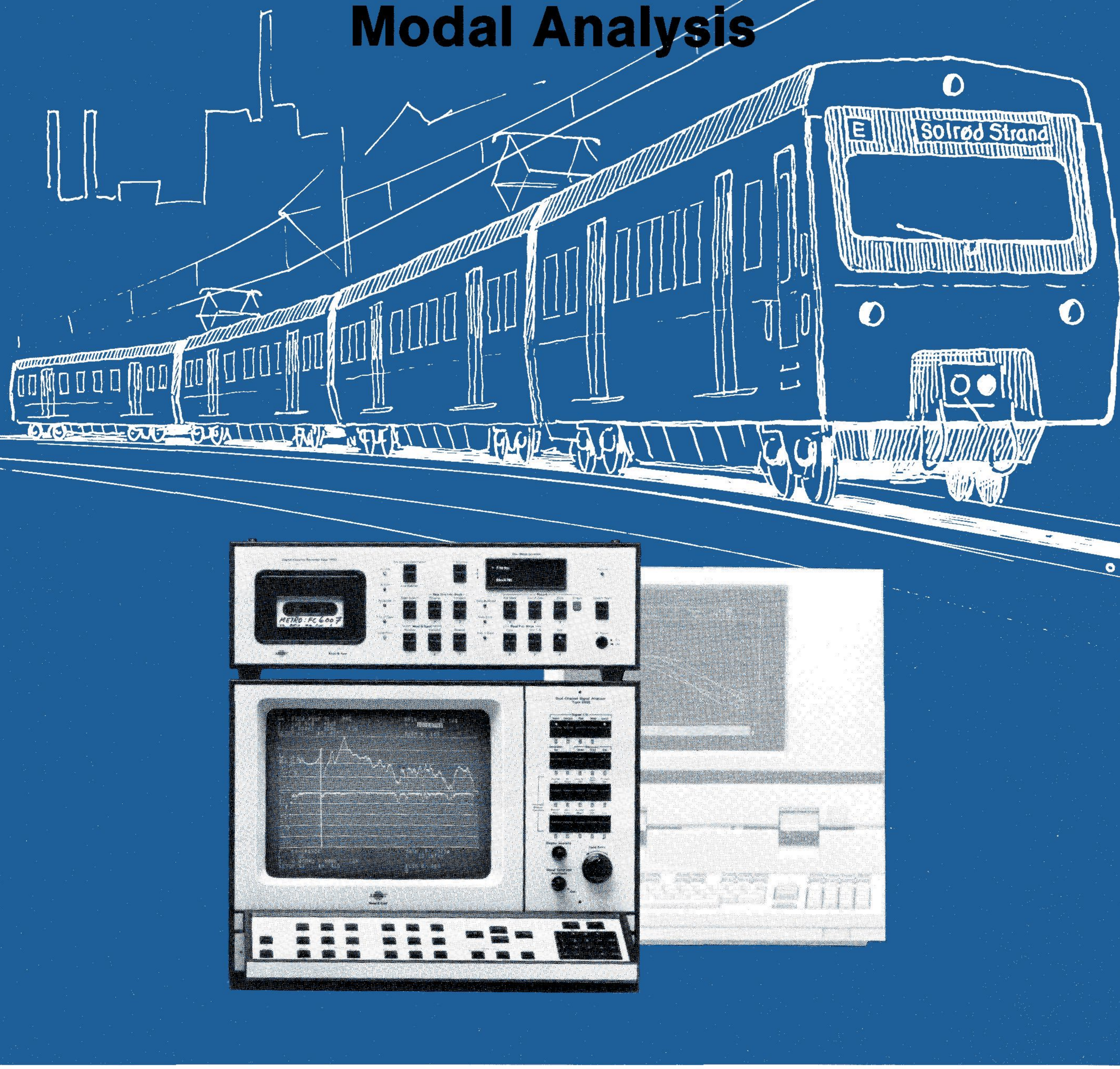
Brüel & Kjær application notes

Dynamic Design Verification of a Prototype Rapid Transit Train using Modal Analysis



Dynamic design verification of a prototype rapid transit train using Modal Analysis

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Abstract

The purpose of the modal test was primarily to verify the natural frequency of the first vertical bending mode for a new train car design.

Measurements were made on two prototype cars plus two similar cars of an older type.

Based on the modal model, the sensitivity of the 1st bending mode to an assumed excitation was estimated, and compared between the four cars.

The change in frequency for the 1st bending mode due to adding payload (passengers) was then predicted by simulation.

This Application Note presents the use of Modal Analysis, by the FRF method, to verify the dynamic design of a prototype rapid transit train, with a brief review of the theory of Modal Analysis as an Appendix.

Introduction

The design and construction of modern mechanical structures is in many cases a process of putting together a set of individually optimized components, and rarely results in an optimal solution.

A typical example is vehicle design. Improved static design tools, implemented on digital computers, have resulted in lighter structures carrying more payload. Improved materials have also led to lighter constructions, which together with increased propulsion power and sharpened requirements with respect to the environment, reliability etc., inherently contain potential ergonomic problems in terms of fatigue, noise and vibration.

The need for dynamic considerations during the design stage was recognised rather early for rotor designs. Here, critical speed problems experienced during turbine operation and gun barrel drilling, together with torsional vibration problems in reciprocating engines, led to the development of schemes for safe dynamic designs.

Today, more and more products are designed using an integrated process. Many types of mechanical devices and

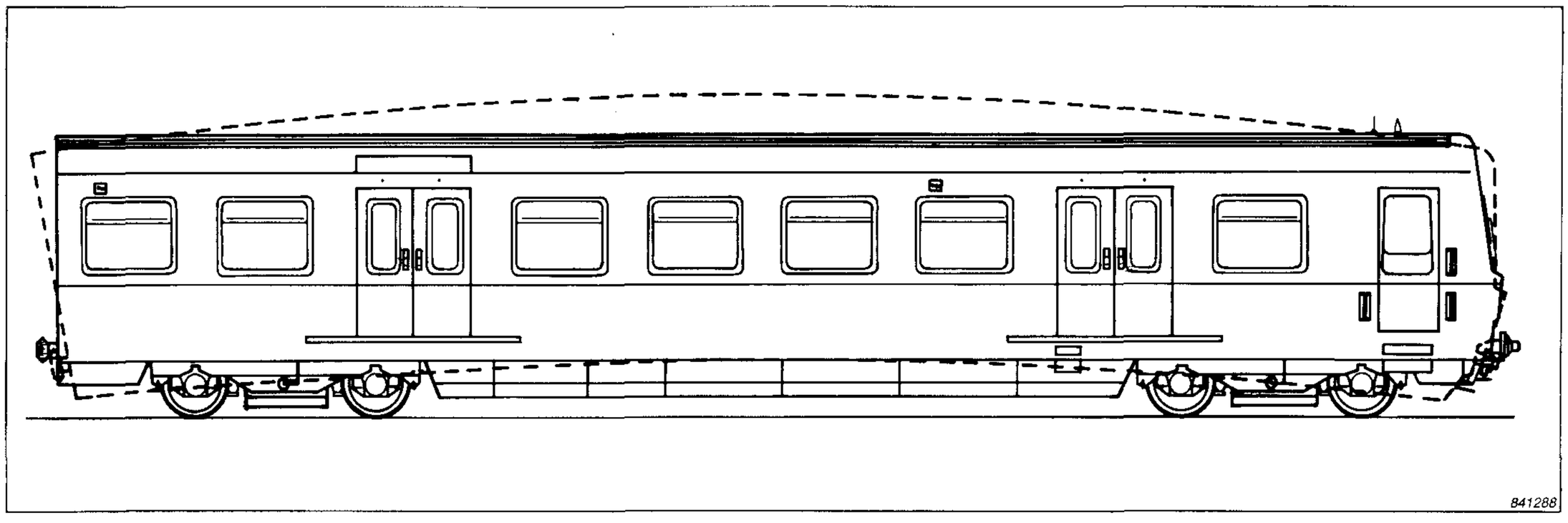

Fig. 1. Rapid Transit Train Type FC car illustrating the 1st bending mode

structures which are expected to be subjected to dynamic forces are designed using dynamic modelling. Some examples are:

— aircraft and spacecraft
— automobiles and trains
— ships and offshore structures
— buildings, dams and nuclear reactors
— engine parts
— business machines and disc drives
— sporting equipment
— household appliances

Besides being used for design purposes, systems can also be modelled to give improved understanding and to allow simulation.

The modelling of systems is generally accomplished by assuming linearity and establishing a set of differential equations. These equations can be solved mathematically to yield the inherent dynamic characteristics or the system response to assumed external forces and boundary conditions. For mechanical systems, the analytical model is implemented in one of 3 ways:

- Continuous Model.
  — using wave equations with boundary conditions. Not applicable for practical complex structures
- Lumped parameter Model.
  — using a very coarse parameter description of idealised discrete elements
- Finite Element Model.
  — using a detailed parameter description of continuous elements

The last two principles have been computerised, and are very useful in early design stages. The resulting model may in turn be used for static analyses, Modal Analysis and for the simulation of response.

As an alternative to the analytical modelling, mathematical models can also be created based on experimental Modal Analysis, yielding:

- Modal Model.
  — using modal parameters; frequency, damping and mode shape

Background

Modal Analysis is the process of determining the modes of vibration. In mathematical terms, the modes of vibration are the eigenvalues and eigenvectors of the system equations. In physical terms, the modes represent the natural frequencies together with the associated deformation shapes.

The natural frequency, together with the damping and mode shape for each mode constitute the **modal parameters**. These parameters represent a complete dynamic description of the structure.

Knowledge of the modal parameters is by itself very useful since it shows at

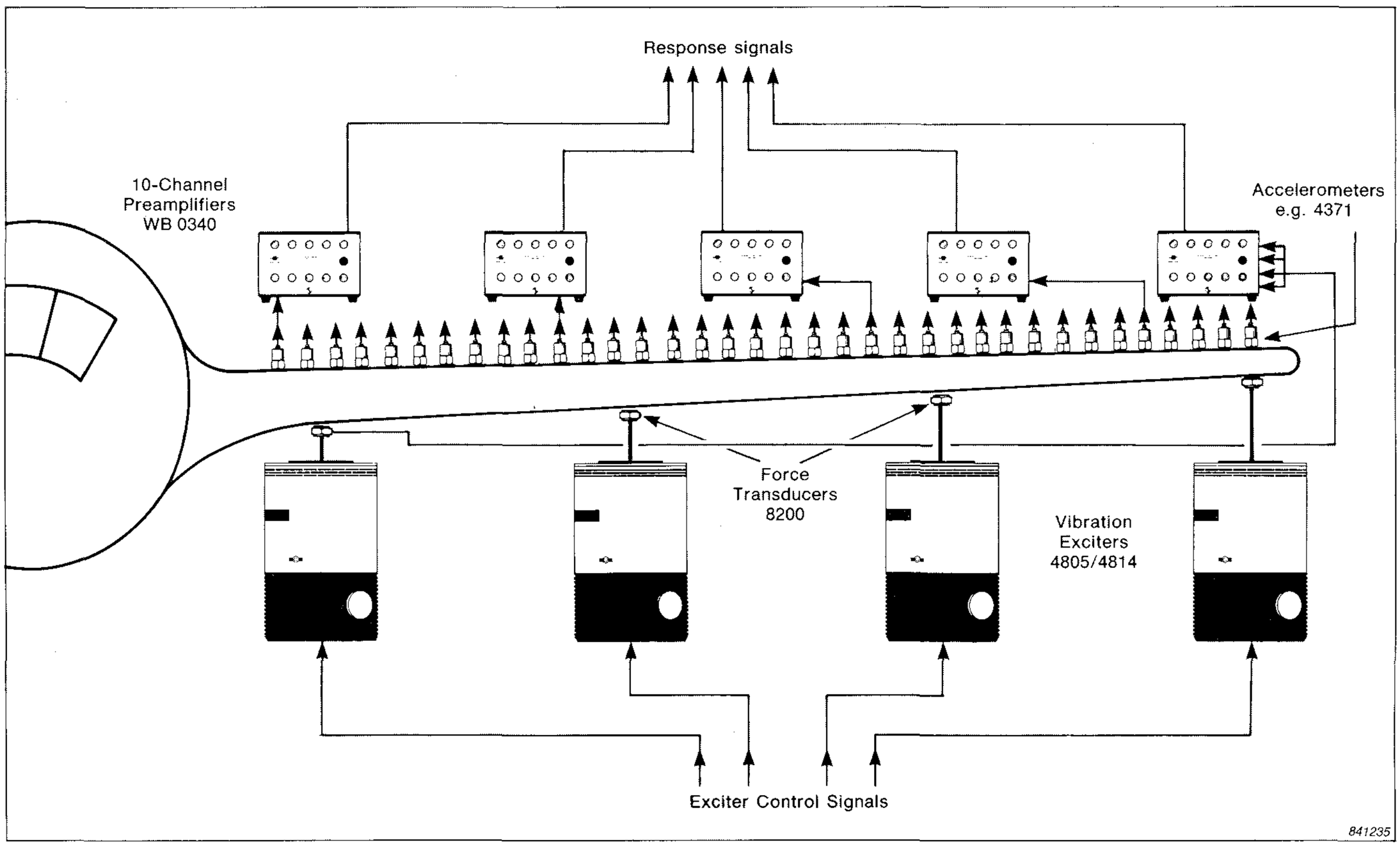

Fig. 2. Part of a typical instrument configuration used for Normal Mode Testing

which frequencies the structure can be excited to resonance. The associated mode shape represents the relative deflections over the structure. This information is often sufficient to suggest structural modifications which will change the dynamic behaviour in a desired direction.

From the modal parameters, a mathematical model for the structure can be created. This model consists of a set of independent equations of motion, each resembling a Single Degree of Freedom (SDOF) system, which in turn can be solved without difficulty.

Apart from the analytical approaches, Modal Analysis can also be accomplished experimentally. Basically two somewhat different methods are available:

- Normal Mode Testing (NMT)
- Frequency Response Function Method (FRF Method)

For both methods the structure is stimulated by a measurable force or forces, and the resulting response observed and related.

Normal Mode Testing

Using NMT, one mode at a time is isolated and the associated modal parameters determined. A number of vibration exciters are distributed over the structure which simultaneously provide excitation forces, and the excitation wave form is sinusoidal. The isolation of a single mode is achieved when the excitation force vector is proportional to the modal deflection for that particular mode. Fig. 2 shows a simplified instrumentation set-up.

The problem with NMT is of a "Catch 22" nature: one needs to know the answer before the question can be asked! A priori knowledge from calculations, or engineering intuition, plus an iterative process does lead to an efficient isolation of each single mode. When a single mode is tuned-in, the system behaves as a Single Degree of Freedom (SDOF) system. The mode shape is determined by measuring the response over the structure, usually by using a large number of accelerometers.

The **undamped** natural frequency is found at the maximum response, and the **damping** determined from the free decay when all the exciters are simultaneously switched off.

Alternatively, using the appropriate force distribution, a "Tuned Sweep" is performed. The quadrature versus coincident response is plotted (Nyquist plot), and the frequency, damping and amplitude determined graphically.

The NMT technique requires a high degree of skill (and art) and a large number of sophisticated instruments. The method produces very accurate results even for large complicated structures with many closely coupled modes e.g. aircraft.

The Frequency Response Function Method

Using the FRF method, the structure can be excited by an arbitrary waveform containing energy distributed over the whole frequency range of interest (broad band testing). All the modes are excited simultaneously and contribute to the observed response. The Frequency Response Function is the complex ratio between the response and excitation spectra.

An FRF is measured at all the points and in all the directions of interest. The measurements need not be made simultaneously, and are generally performed sequentially. With the advent of 3rd and 4th generation Dual-Channel FFT Analyzers and powerful computer programs, with very user-friendly interfaces, the FRF method has become increasingly popular due to shorter test times and reasonably priced equipment. This method is now adopted throughout industry and research institutions.

Rapid Transit Train Modal Testing

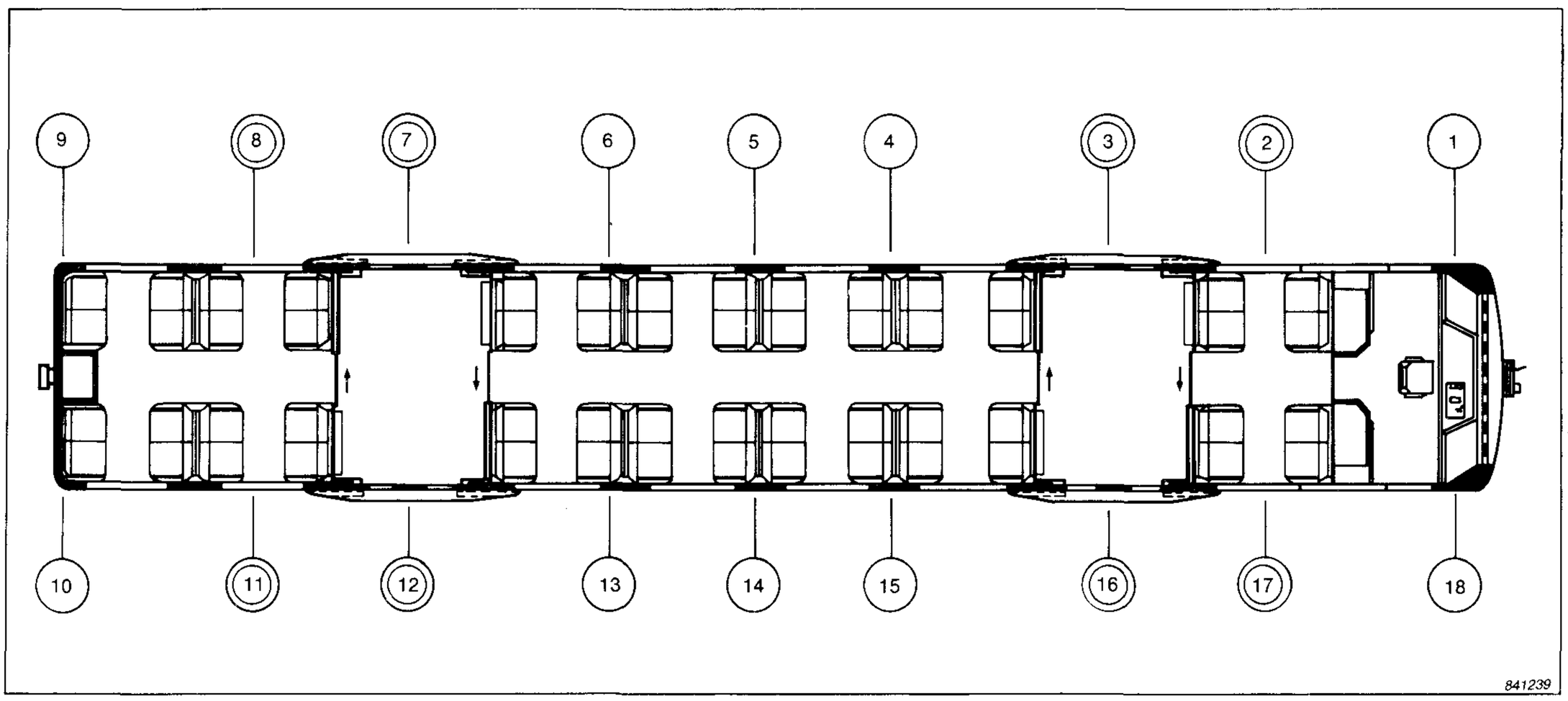

Fig. 3. FC type car showing the measurement points

Purpose and Scope

The purpose of the modal test was primarily to verify the natural frequency of the first vertical bending mode for a new train car design.

As the modal density of a train car is high, it was necessary to identify a number of modes around the 1st bending mode to secure correct identification.

Dynamic Design Criterion

The major source of vertical excitation of a train car is a combination of the elasticity of the rail, non uniformities in the track surface, and the discontinuity of the rail joints. This excitation is rather broad banded, but the bogie suspension system is designed to yield an efficient isolation at frequencies higher than 8—10 Hz, and to control the rigid body motions. However, impacts and irregularities in the rail track do create forces at higher frequencies exciting the elastic motion of the car. This excitation can be seen as a symmetric force distribution entering the carriage through the bogie connections.

If the wheels are not perfectly round, some excitation at the wheel frequency ($\approx$ 10 Hz) and at the higher harmonics can be expected.

A pure, symmetric excitation can only excite symmetric modes (an illustration explaining this statement is given in Fig. 9). Thus the lowest symmetric elastic mode of the carriage, the 1st bending mode constitutes the primary elastic contribution to the dynamic behaviour of the car.

In this case the design criterion is that the natural frequency for the 1st bending mode should be higher than 10 Hz.

The calculations for the car were rather simplified, and based on the "Redundant Frame Method" (Biech, 1924).

Testing Procedure

Measuring Object

All the four train cars considered are very similar, and are:

1. FC new Driving trailer, 34,5 ton
2. MC new Motor coach, 46,3 ton
3. FS old Driving trailer, 27,8 ton
4. MU old Motor coach, 42,8 ton

Builders:	Scandia-Randers Ltd., Denmark
Owners:	DSB (Danish National Railroads)
Consultants:	Ødegaard and Danneskiold-Samsoe K/S

Fig.3 shows a type FC train car.

Measurement Condition

The measurements were accomplished with the vehicles placed on rails in the DSB maintenance shop, Taastrup, Denmark. The cars were fully outfitted, but without payload.

Measurement Positions

In order to make an accurate identification and to obtain a useful dynamic modal model, a relatively high number of measuring points were chosen (see Fig.3). Since the purpose was to identify the 1st bending mode, measurements were made in the vertical direction only.

Instrumentation

Fig.4 shows the instrumentation. The numbers refer to B&K products. Further details of these products are given in Ref. [7].

Excitation

Due to the anticipated non-linear behaviour of the suspension system, Random Noise was chosen as the excitation waveform since this gives the best linear approximation for non linear systems (Ref. [1]).

The waveform was generated by the Dual Channel Signal Analyzer Type 2032, producing a flat spectrum with a frequency distribution limited to the selected measurement range.

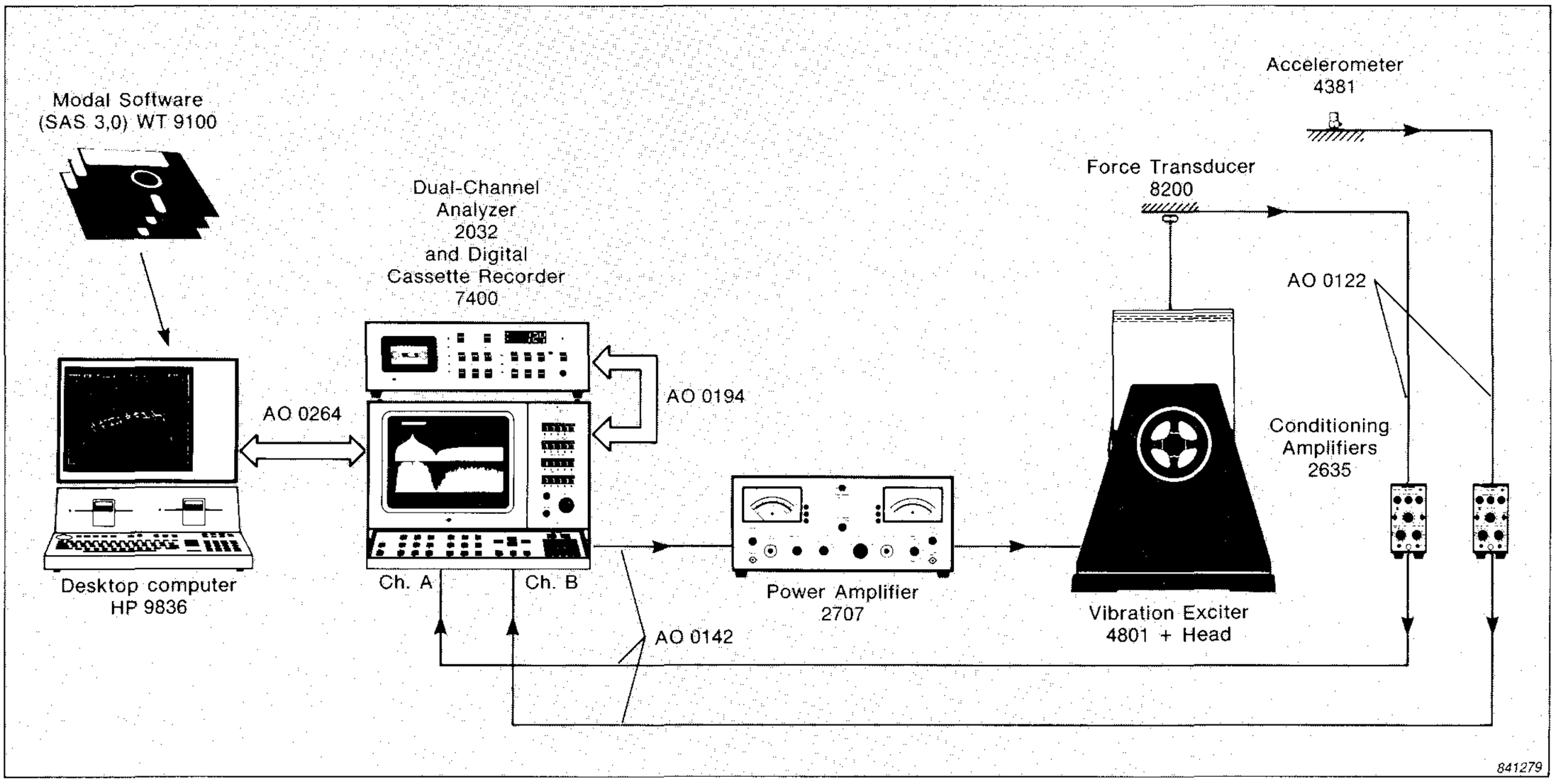

Fig. 4. The instrument set-up used to make the measurements and Modal Analysis

The random force was implemented by feeding the generator signal to the Power Amplifier Type 2707, driving the Vibration Exciter Type 4801 fitted with Exciter Head Type 4814. Connection to the structure was by a stinger, a thin (⌀ 3 mm) nylon rod, axially stiff and laterally flexible. The purpose of a stinger is to prevent rotational and lateral coupling between the exciter and structure. The force input to the car was measured by the Force Transducer Type 8200, screwed directly into the structure.

The Excitation DOF* was chosen as a point at one corner of the car and applied in the vertical direction. The asymmetric excitation position ensured that all modes were excited.

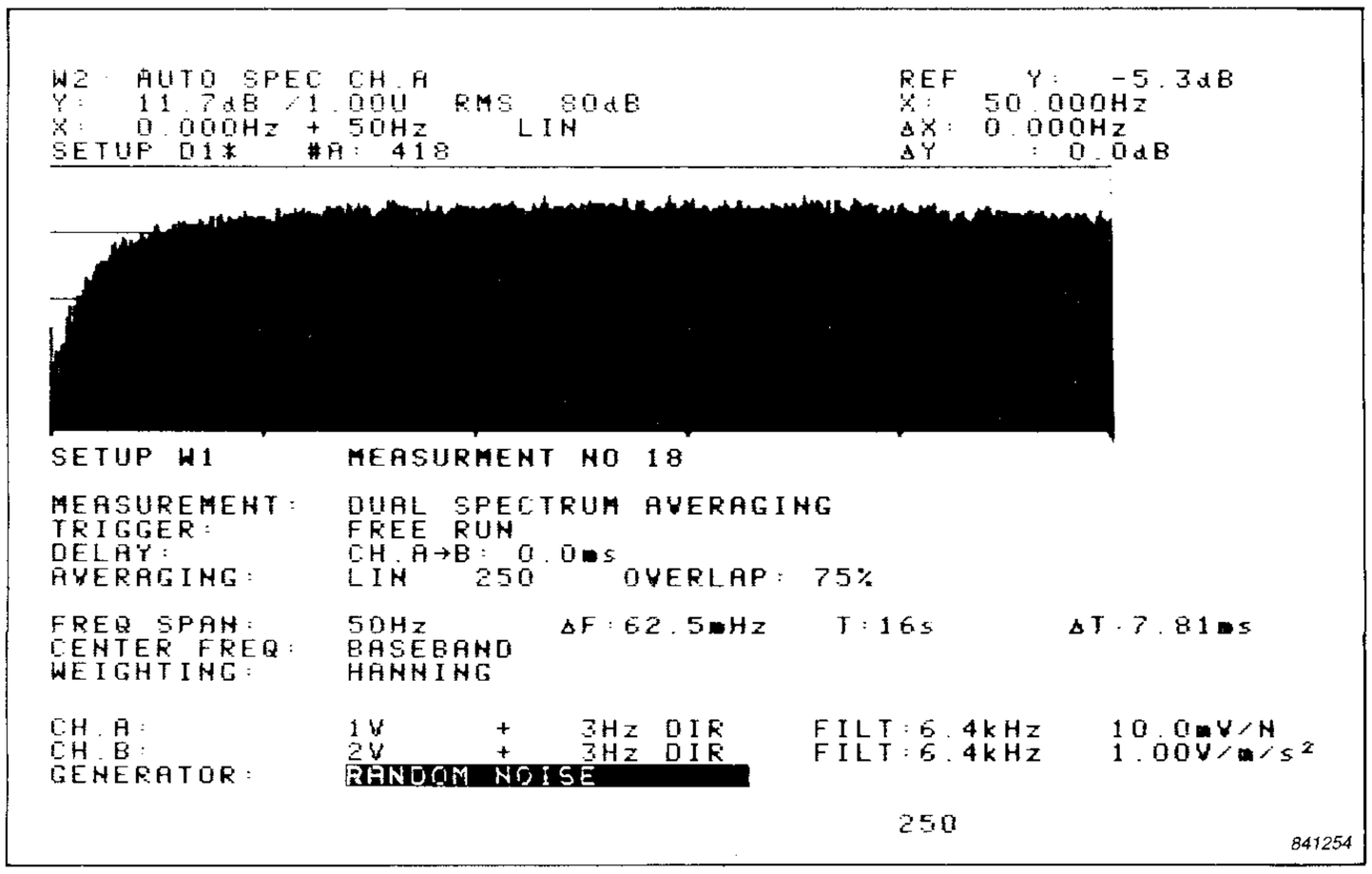

Fig. 5. Excitation Spectrum with measurement parameters as displayed by the Dual Channel Signal Analyzer Type 2032

Response

The vibration response was measured using a Piezoelectric Accelerometer Type 4381. The accelerometer was mounted using the magnet UA 0642. The magnet, applied at a smooth surface provided a rapid and, for the frequency range, good mounting.

Signal Analysis

The transducer signals were conditioned in the Charge Amplifiers Type 2635 and fed to the Dual Channel Signal Analyzer Type 2032.

The measurement frequency range was 0 to 50 Hz. The 801 line transformation gave a frequency resolution of 0,0625 Hz.

First the force spectrum was checked. Fig.5 shows the excitation spectrum together with all the measurement parameters of the analysis. The spectrum is flat from approx. 2,5 to 50 Hz. The force level achieved by the setup was approx. 200 N RMS.

FRF-Measurement

The measurements were accomplished during normal working hours in the maintenance shop. The activity in the shop created some mechanical background noise at the output. Selecting the FRF estimator (Ref. [1]),

$$H_1 = \frac{G_{XF}}{G_{FF}}$$

where G_{XF} is the average Cross Spectrum between response and excitation
G_{FF} is the Autospectrum of the excitation

and employing an appropriate number of averages, the influence from the background noise was removed.

Fig.6 shows an example of a measured FRF together with the Coherence Function. The Coherence Function implies a very good linear relationship between the response acceleration and the excitation force in the full frequency range. The observed drops in coherence at low frequencies are explained by the low excitation force in this range, giving a low signal-to-noise ratio, and from the expected non-linear behaviour of the suspension system.

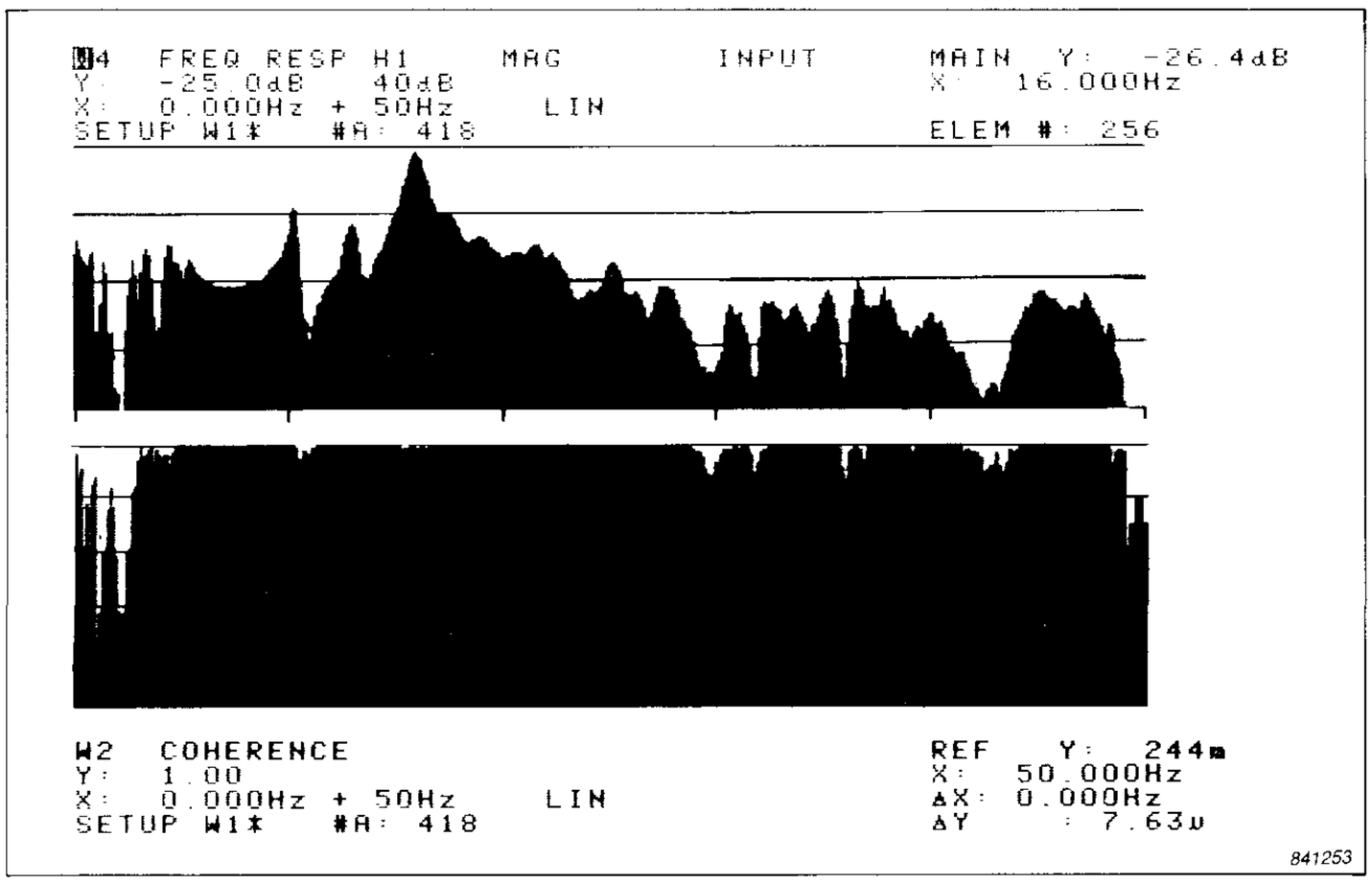

Fig. 6. Measured FRF and Coherence Function as displayed by the Dual Channel Signal Analyzer Type 2032

Drops in coherence at the antiresonances were expected, as antiresonances

* DOF = Degree of Freedom, is motion at a point in a particular direction.

ances represent frequencies at which a structure does not response to excitation, giving a very poor signal-to-noise ratio.

Data Aquisition

An FRF measurement was made for each specific DOF. The accelerometer was sequentially mounted, and 250 spectra with 75% overlapping were averaged together. After completion of the averaging, the FRF was transferred to the computer, labelled (with DOFs, time, etc.), and stored on a 5¼″ floppy disc. A number of measurements, together with complete measurement documentation, were dumped to the Digital Cassette Recorder Type 7400 for reference and security.

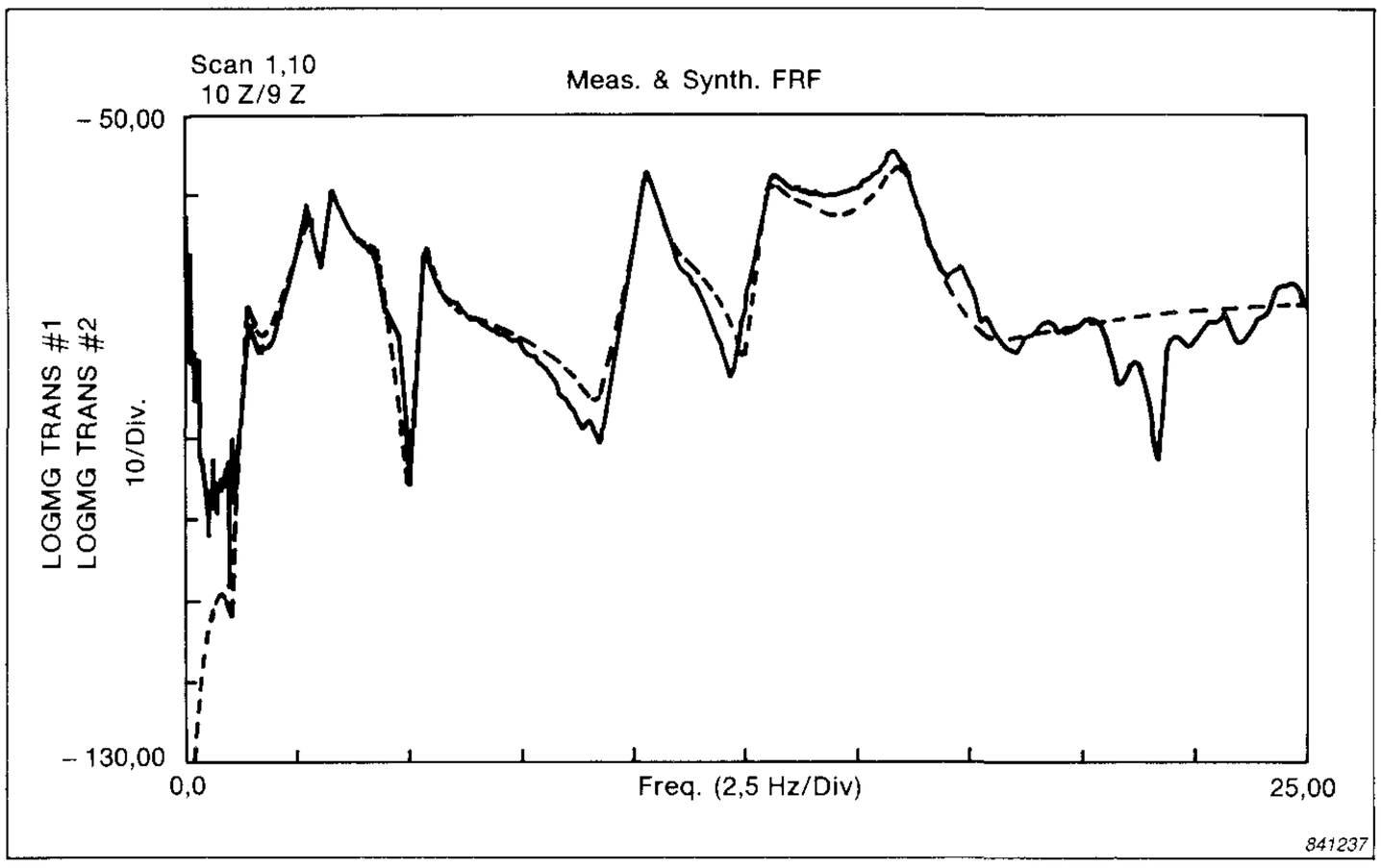

Fig. 7. Plot of the measured FRF and synthesized FRF (dotted line)

Modal Analysis

For data management, modal parameter estimation, mode shape animation and documentation, the Brüel & Kjær Structural Analysis System Type WT 9100 was used, installed in a HP 9836 computer (Ref. [5]).

Parameter Estimation

The modal parameter estimation was done by curve fitting. First one typical FRF measurement was recalled and displayed on the computer screen.

A frequency band around each mode was then specified by a cursor pair. The curve fitting method was selected and the modal parameters were estimated by the system. Based on the estimated parameters an FRF in the cursor band was synthesized and displayed together with the measured FRF for evaluation of the fit.

While the typical FRF measurement was being fitted, an AUTOFIT table was automatically updated. Fig.7 shows an FRF measurement together with a synthesis. An AUTOFIT table, specifying the curve fitting bands and procedures, is shown in Table 1.

By entering an AUTOFIT command all FRF measurements are automatically recalled to the computer, and the modal parameters are estimated based on the AUTOFIT table specifications. The result is a complete set of modal parameters.

Documentation

The system provides a wide variety of documentation facilities. The most powerful of these is the real time mode shape animation. Here the graphics computer display animates the modes of vibration in slow motion. Printer/plotter outputs are also available.

MODE	LOW CURSOR	HIGH CURSOR	FIT METHOD
1, 6 +2	0,94	6,94	MDOF-LSCEXP
7	9,5	11,13	SDOF-POLY
8, 9 +2	12,06	14,75	MDOF-LSCEXP
10	15,31	16,94	SDOF-POLY

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Table 1. AUTOFIT table for car no. 1

Mode	Frequency $=\omega_d/2\pi$ (Hz)	Damping $=\zeta$ (%)	Damping $=\sigma/2\pi$ (Hz)
1	1,43	4,58	0,07
2	2,76	5,64	0,16
3	3,34	4,93	0,16
4	4,34	5,21	0,23
5	5,39	2,27	0,12
6	6,58	5,27	0,35
7	10,36	1,57	0,16
8	13,12	2,04	0,27
9	14,05	3,41	0,48
10	16,04	2,26	0,36

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Table 2. Frequency and damping for the first ten modes of car no.1

Results

For train car no.1, Table 2 shows the frequency and damping for 10 modes of vibration. A number of the mode shapes are shown in Fig.8. Identification of 1st bending mode is now

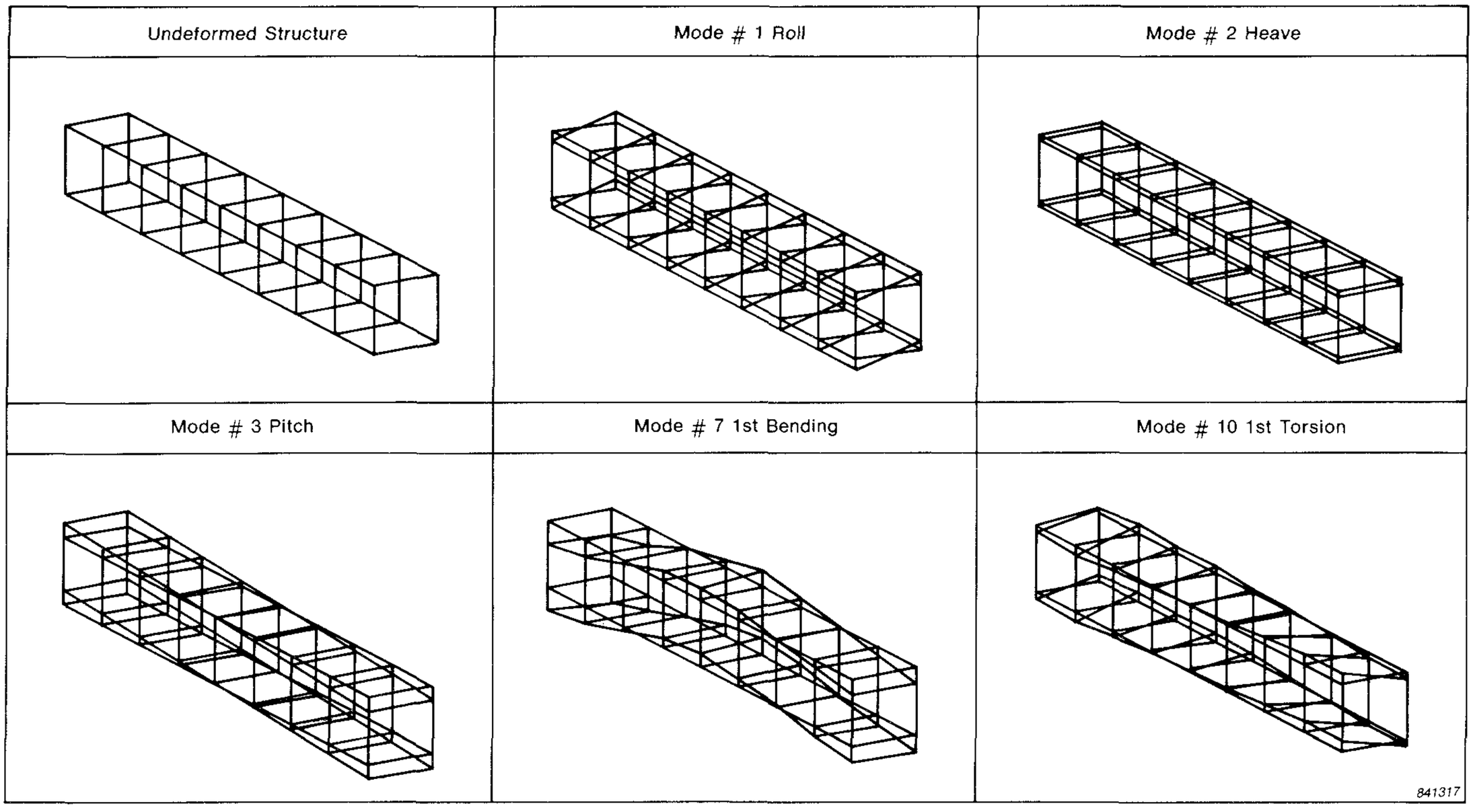

Fig. 8. The undeformed structure and 5 different mode shapes

rather obvious. The frequencies and damping for this mode are shown in Table 3 for the four cars.

Car no.	Frequency (Hz)	Damping (%)
1	10,4	1,6
2	12,2	1,9
3	12,4	2,2
4	10,8	4,3

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Table 3. Frequencies and damping for the first bending mode of the 4 cars

Conclusion

The primary purpose of the modal test was to identify the 1st vertical bending mode, which was well accomplished. The results show that the design criterion – Natural frequency of the first bending mode shall be greater than 10 Hz – is achieved.

Beyond the primary task, a number of the higher elastic modes were also identified. Due to non-linearities the rigid body modes were identified with some difficulty. For a linear system a mode shape is a free vibration property and is independent of the excitation position. The non-linear effects showed up as distorted mode shapes where the deflections seems highest around the excitation point.

Advanced Applications

Bending Mode Sensitivity

The result of the Modal Analysis can be used for a closer inspection of the properties of the 1st bending mode for the individual cars.

Generalized Coordinates

The most important property of the mode shapes is their mutual orthogonality. This property is used in a linear transformation, the Modal Transformation:

$$\{x\} = [\phi]\{q\} \quad [1]$$

where $\{x\}$ are the displacements in physical coordinates
$\{q\}$ are the generalized coordinates
$[\phi] = [\{\phi\}_1\ \{\phi\}_2\ \ldots\ \{\phi\}_m]$ is the modal matrix where the columns are the mode shapes

Using the modal transformation, the equations of motion become uncoupled, and the number of coordinates is reduced to the number of modes.

Now we want to study the properties of the 1st bending mode, and the problem can be reduced to a single DOF system.

By only considering the 1st bending mode we can examine the car in generalized coordinates using the associated mode shape for a transformation.

$$\{x\} = \{\phi\}_r \cdot q_r \quad [2]$$

where $\{x\}$ is the physical coordinate
$\{\phi\}_r$ is the mode shape scaled to unit modal mass
q_r is the generalized coordinate

and the equation of motion is:

$$\ddot{q}_r(t) + 2\sigma_r \dot{q}_r(t) + \omega_{0r}^2 q_r(t) = Q_r \quad [3]$$

where σ is the decay rate representing damping
ω_0 is the undamped natural frequency
Q is the generalized force

All the parameters in this equation of motion represent the output from the Modal Analysis.

Generalized Force

The transformed excitation forces are the Generalized Force:

$$Q_r \equiv \{\phi\}_r^T \{f\} = \sum_{i=1}^{n} \phi_{ir} \cdot f_i \qquad [4]$$

where $\{\phi\}_r$ is the r^{th} mode shape
$\{f\}$ is the excitation force vector

The Generalized Force Q_r has a very simple physical interpretation, since it expresses the ability of a given force vector $\{f\}$ to excite a particular mode $\{\phi\}_r$.

If we assume that the vertical excitation forces on the car are symmetrically distributed, we may graphically evaluate the Generalized Force (Fig. 9), demonstrating that asymmetric modes cannot be excited by symmetric excitation as the Generalized Force becomes zero.

To compare the 1st bending mode sensitivity for the four cars we can calculate the Generalized Force. Assuming an arbritarily chosen force vector of 1000 N at each of 4 DOFs around each bogie/car connection (outlined by double circles in Fig. 3), the results are shown in Table 4.

Car no.	Generalized Force (N)
1	Q = 43
2	Q = 35
3	Q = 38
4	Q = 65

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Table 4. Generalized Forces for the 4 cars

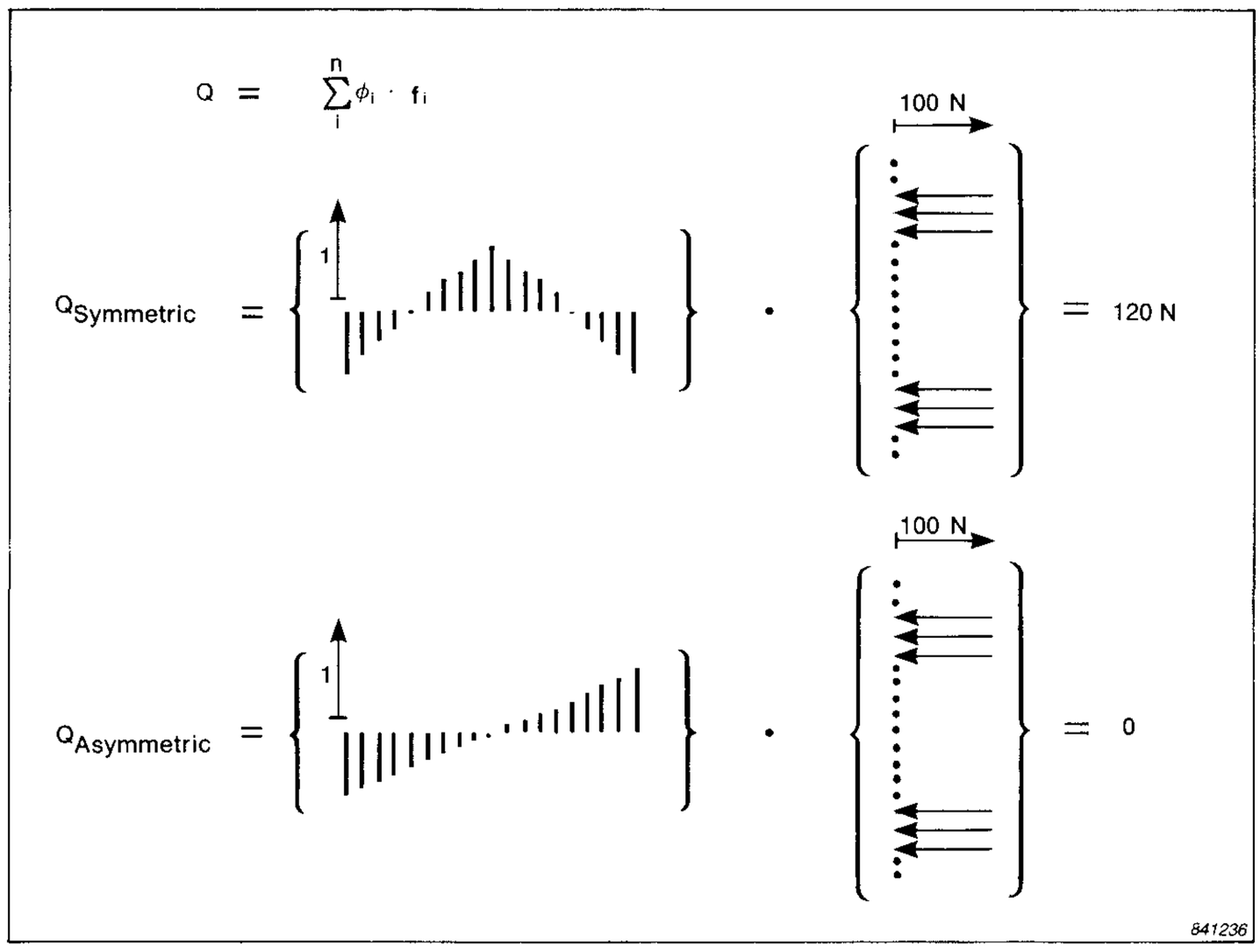

Fig. 9. Generalized Forces for both symmetric and asymmetric modes using symmetric excitation

If we assume a 10 Hz harmonic excitation (corresponding to the wheel frequency) with the same force distribution as given above, we can calculate the modal or generalized response. The results are shown in Table 5.

Car no.	Modal Response Magnitude (mm)
1	q(10 Hz) = $13{,}30 \times 10^{-2}$
2	q(10 Hz) = $1{,}8 \times 10^{-2}$
3	q(10 Hz) = $1{,}78 \times 10^{-2}$
4	q(10 Hz) = $8{,}48 \times 10^{-2}$

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Table 5. Generalized or Modal response for 10 Hz excitation

Generalized Response

The generalized response to an actual excitation is calculated by transforming to the frequency domain:

$$q(\omega) = \frac{Q(\omega)}{-\omega^2 + 2j\sigma\omega + \omega_0^2} \qquad [5]$$

where $q(\omega)$ is the spectrum of the displacement in the modal coordinate,
$Q(\omega)$ is the spectrum of the Generalized Force.

Discussion

Comparison of the dynamic properties of the four cars with respect to an assumed excitation can be based on the Generalized Force and the generalized response for the 1st bending mode. From this, it appears that the new motor coach is less sensitive than the old type. The new driving trailer appears to be more sensitive than the old design. This is particularly the case with respect to the wheel frequency excitation due to a lower natural frequency and damping. However, the damping is significantly underestimated due to the bogie damper adjustments and due to the non-linear damping mechanism.

Physical Response

When we have calculated the response (displacement) in modal coordinates, the results can be transformed back to physical coordinates using equation [1].

In our example, where we only consider one mode, the transformation is reduced to [2].

The interpretation is that the physical deformation is equal to the mode shape multiplied by the generalized response. Alternatively, that the response at a specific point is equal to the modal deflection at that point times the generalized response:

$$x_i = \phi_{ir} \cdot q_r \qquad [6]$$

Damping

For the response estimation [5], the damping term is included. The modal damping is generally the least accurately estimated modal parameter.

Looking at the modal damping for the 1st bending mode for the four cars we observe a relatively high difference in damping. This difference can be explained from the mode shapes and boundary conditions.

As the 1st bending mode frequency is three times higher than the highest rigid body mode, the car may be considered as a free/free beam. However, the node points for this mode are not at the support points, thus damping forces are introduced from the suspension.

The damping force depends on :

- mode shape
- bogie design/condition
- vibration level (non linear)

The two last terms were not controlled during the test, and thus damping deviation must be expected.

Simulation of Loading the Train with Passengers

Structural Dynamics Modification

When the trains are in operation, and loaded with passengers, the dynamic characteristics will change.

The dynamic modification can be simulated using the modal parameters.

The equation of motion can be written:

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f\} \qquad [7]$$

where the vectors;

$\{x\}$, $\{\dot{x}\}$, $\{\ddot{x}\}$ are the displacement, velocity and acceleration over the structure

and the matrices;

[m], [c], [k] are the spatial parameters representing the mass, damping and stiffness in the structure

Equation [7] represents a set of second order differential equations. The spatial parameters are unknown and cannot be measured in the experimental analysis. Using the modal transformation [1], the equations are uncoupled in the generalized coordinates.

By also using a specific scaling of the mode shapes – Unit Modal Mass Scaling:

$$[\phi]^T [m][\phi] = [I]$$

the equation of motion reduces to:

$$[I]\{\ddot{q}\} + [2\sigma]\{\dot{q}\} + [\omega_0^2]\{q\} = [\phi]^T\{f\} \qquad [8]$$

with the advantages that:

- the equations are uncoupled and may be solved individually
- all the parameters can be measured through Modal Analysis
- the number of unknowns is reduced to the number of modes

Introducing modifications to the system parameters; [Δm], [Δc] or [Δk], these modifications are transformed to generalized coordinates by:

$$[\phi]^T[\Delta][\phi]$$

and added to the original coefficient matrix in equation [8]. The eigenvalue solution to the new equation yields the new frequencies, damping and mode shapes.

As an example, an addition of masses [Δm] to a system can be simulated by solving the determinant equation:

$$\left|\left[[I] + [\phi]^T[\Delta m][\phi]\right] s^2 + [2\sigma] s + [\omega_0^2]\right| = 0 \qquad [9]$$

Equation [9] represents a polynomial in s of the order $2 \cdot m$, where m is the number of modes. The roots of the polynomial represent the new natural frequencies and damping.

MODIFICATION HISTORY			
No.	**Type**	**Amount**	**DOF**
1	mass	140,00 kg	1
2	mass	280,00 kg	2
3	mass	280,00 kg	3
4	mass	280,00 kg	4
5	mass	280,00 kg	5
6	mass	280,00 kg	6
7	mass	280,00 kg	7
8	mass	280,00 kg	8
9	mass	140,00 kg	9
10	mass	140,00 kg	10
11	mass	280,00 kg	11
12	mass	280,00 kg	12
13	mass	280,00 kg	13
14	mass	280,00 kg	14
15	mass	280,00 kg	15
16	mass	280,00 kg	16
17	mass	280,00 kg	17
18	mass	140,00 kg	18

Σ = 4480 kg

Table 6. The modification history

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Substituting each root p:

$$\left[\left[\lceil I \rfloor + [\phi]^T[m][\phi]\right] p^2 + \lceil 2\sigma \rfloor p + \lceil \omega_0^2 \rfloor\right]\{q(s)\} = 0 \quad [10]$$

yields the solution vector $\{q\}_p$, which represents the new mode shape. The modal coordinates used to calculate this new mode shape are then transformed to physical coordinates using equation [1].

In the SAS 3.0 / WT 9100 software package, the subset SDM 3.0 simulates mass, damping and stiffness modifications using this technique.

The computational time for a modification simulation is typically 10 seconds.

Car no.	Before (Hz)	After (Hz)	Frequency shift (Hz)
1	10,36	10,08	0,28
2	12,15	11,81	0,34
3	12,35	11,90	0,45
4	10,84	10,43	0,41

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Table 7. Predicted frequency shifts derived by simulating structural modifications

With the purpose of predicting the shift in frequency of the 1st bending mode, a structural dynamic modification was simulated. 70 kg of mass was assumed at each passenger seat. The masses were distributed to the measured DOFs as shown by the modification history in Table 6. The predicted frequency shifts for each train car are shown inTable 7.

Even though the predicted frequency shifts are rather small, it can be seen that for car no. 1, the new driving trailer, the frequency of the 1st bending mode is approaching the critical 10 Hz design criterion.

Conclusion

This Application Note shows an example of the experimental dynamic analysis of a complex structure, with user-friendly, integrated instrumentation.

Time consumption was four hours for set-up and trial measurements, followed by three hours for each of the four train cars tested.

The analysis verified the dynamic design of the prototype rapid transit trains, and produced a mathematical model of the structure. This model was employed both for estimating the vertical bending sensitivity to assumed excitation, and for predicting what effects loading the cars with passengers has on the dynamic properties.

Appendix

Introduction to Modal Analysis

This appendix presents an intuitive rather than mathematical introduction to the theory of Modal Analysis.

Modal Behaviour

The dynamic response of a structure, caused by an arbitrary forcing function, is the sum of a discrete set of *independent, well defined motions* (Fig. 10 a). These motions are the **modes**, and each mode is described by:

- The mode shape
- The modal frequency
- The modal damping

Each mode shape describes the relative deflections over the structure. A mode shape may be classified as normal or complex. For a normal mode shape, all the points are moving either in phase, or 180° out of phase. This means that all the points are arriving at maximum deflections, or passing the undeformed state, simultaneously. The normal mode may be considered as a standing wave in a structure.

Normal modes are generally found *in structures with either evenly distributed or low damping. Structures* with either high or unevenly distributed damping often exhibit complex modes.

For the complex mode, the phase between motions over the structure may be arbitrary. The complex mode shape behaves more like a propogating wave, and nodes are not stationary.

For a continuous structure, a mode shape is a continuous spatial function (Fig. 10 b). It is generally discretely sampled and is presented and treated as a vector $\{\psi\}$. The mode shape vector contains the modal deflection at selected points and directions ψ_i.

The modal frequencies and damping have been given various names: roots, eigenvalues, pole location, resonance, natural frequency, etc. A very *convenient way to communicate* modal frequency and damping is by the pole location:

$$p = -\sigma + j\omega_d \quad [11]$$

where p is a complex number (complex eigenvalue) composed of a real part,

$$\sigma = \zeta \cdot \omega_0 \quad [12]$$

representing the decay rate σ of the free vibration, and equal to the fraction of critical damping ζ times ω_0 the

undamped natural frequency. σ also represents half of the 3 dB bandwidth of the resonance in the frequency domain.

The imaginary part of p,

$$\omega_d = \omega_0\sqrt{1 - \zeta^2} \qquad [13]$$

is the damped natural frequency, and represents the oscillation frequency of the free vibration, as well as the frequency at which the system can be excited to resonance (Fig .10 c).

The set of mode shapes and the pole locations, the **modal parameters**, constitute a complete dynamic description of a structure.

A continuous structure has an infinite number of modes. For the dynamic description, a truncated model containing only the modes present in the frequency band of interest can be used and gives sufficient accuracy.

Modal Analysis is the process of determining the modal parameters.

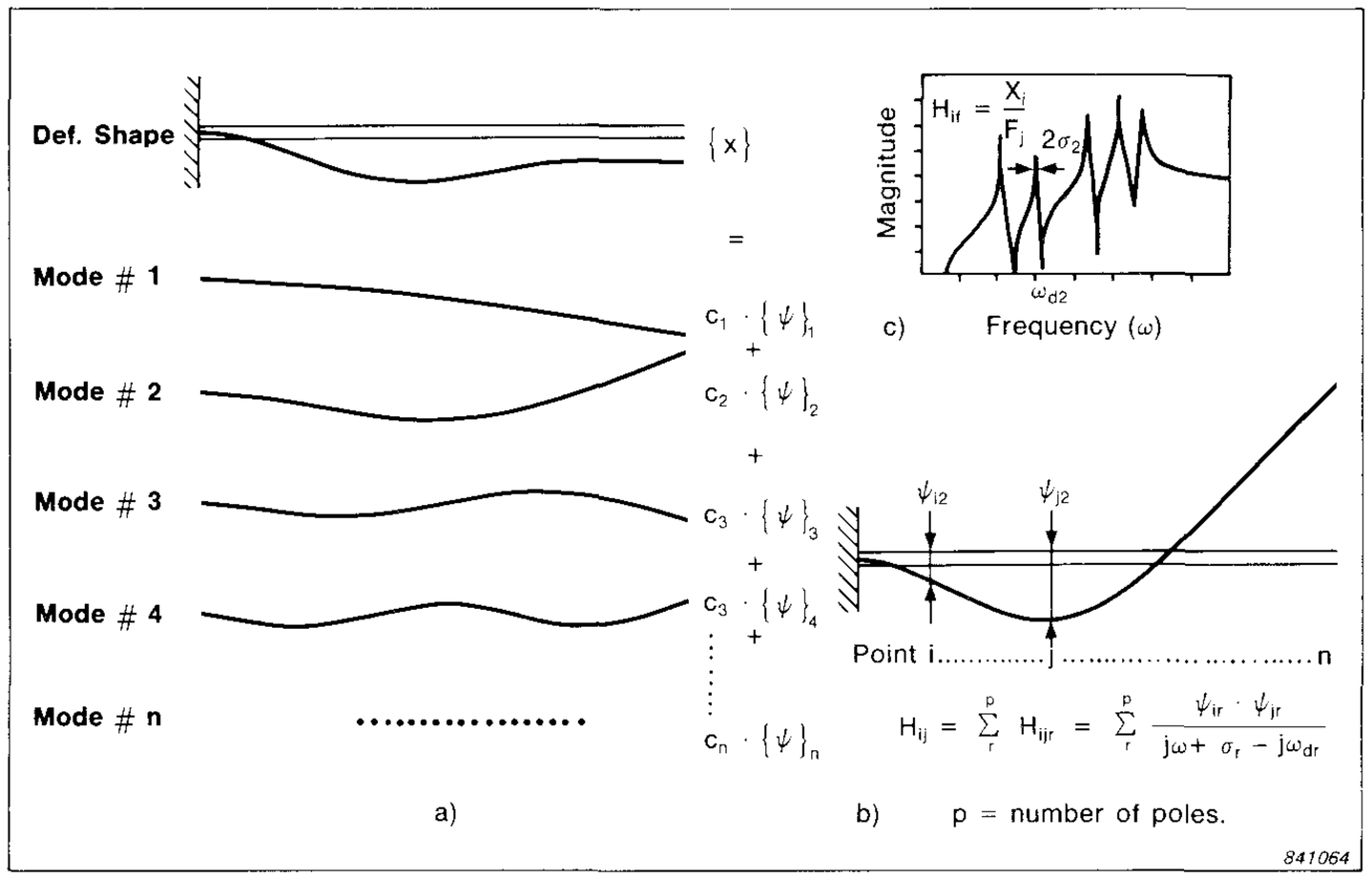

Fig. 10. Modal behaviour

The Frequency Response Function

In experimental Modal Analysis, measurements of the input/output relations are made. Today the most convenient input/output measurement is the Frequency Response Function (FRF).

The FRF is by definition the complex ratio between the output spectrum, measured in DOF* i, and the excitation spectrum in DOF j:

$$H_{ij}(\omega) = \frac{X_i(\omega)}{F_j(\omega)} \qquad [14]$$

The FRF has two indices identifying the response and excitation DOF. It is a function of frequency ω and is complex i.e. it has a real and imaginary part (or magnitude and phase). FRFs are properties of linear systems which:

- do not depend on the type of excitation,
- can be measured with sinusoidal, random or transient excitation,
- if obtained with one type of excitation can thus be used to predict the system response to any other type of excitation,
- can be decomposed into the modal parameters.

FRF Measurements

The definition of the FRF implies that we only need to excite the system by a measurable force, measure the associated response, make transformations to the frequency domain, and calculate the ratio between the two spectra.

In real measurement situations it is not possible to measure the true input and output due to noise. By the term noise we include:

- mechanical background noise
- ambient excitation
- electrical noise in the transducing system
- data acquisition noise
- computational noise
- non-linearities, etc.

Due to the contaminating noise in our data we have to define an estimator for the FRF. Ref. [1] discusses, in detail, the measurement techniques and the estimates for the FRF. These estimates are based on the measurement of the Cross Spectrum G_{XF} and one Autospectrum from the response G_{XX} or the excitation G_{FF}.

$$H_1(f) = \frac{G_{XF}(f)}{G_{FF}(f)} \text{ or } H_2(f) = \frac{G_{XX}(f)}{G_{FX}(f)} \qquad [15]$$

Another useful function based on the same spectra may be calculated;

$$\gamma^2(f) = \frac{|G_{XF}(f)|^2}{G_{XX}(f)\,G_{FF}(f)} \qquad [16]$$

which is the Coherence Function. The Coherence Function expresses the linear relation between response and excitation, and is used for evaluating the quality of the measurements. Once again Ref. [1] discusses the Coherence Function in detail.

Alternative Formats

So far we have only discussed FRFs in terms of response and excitation. Dynamic response can be expressed in terms of displacement, velocity or acceleration. As a result, the FRF can be in three different formats:

Compliance:

$$H(\omega) = \frac{X(\omega)}{F(\omega)}$$

Mobility:

$$M(\omega) = \frac{\dot{X}(\omega)}{F(\omega)} = H(\omega)\cdot j\omega$$

Accelerance:

$$A(\omega) = \frac{\ddot{X}(\omega)}{F(\omega)} = H(\omega)\cdot(-\omega^2)$$

Having measured one of the response functions the others may be calculated. Today most FFT analyzers provide this facility. Currently it is the

* DOF = Degree of Freedom, is motion at a point in a particular direction.

measurement of Accelerance which is most popular due to the convenient application of accelerometers. For analytical work, modelling, and curve-fitting Compliance is normally used. The previously so popular Mobility is now rarely used for Modal Analysis.

The FRF and Modal Parameters

Analytically the FRF can be decomposed into modal parameters, and from a measured FRF the modal parameters may be estimated.

Modal Frequency and Damping

The pole location, i.e. the damping and damped natural frequency, are represented by the resonance frequency and the bandwidth of the resonance peak (see Fig. 10 c).

The pole locations are global properties, i.e. they can be measured anywhere on the structure.

Mode Shape

If we assume low coupling between the modes (i.e. sufficient frequency spacing compared to the width), we can make an intuitional interpretation:

The FRF, $H_{ij}(\omega)$ generally represents the deflection at i per unit force at j. At the resonance for the r^{th} mode the contribution from the other modes is negligible, and now $H_{ij}(\omega_{dr})$ represents the modal deflection in i per unit force in j for mode r alone (see Fig. 11).

Either one response, or one excitation DOF is chosen as a reference. FRFs are measured at all DOFs of interest. Now, the set of H-values measured over the structure for each resonance represent the associated mode shape. Fig. 12 shows a set of FRFs measured along a cantilever beam. The imaginary amplitude of the H along the three resonant frequencies forms the mode shapes.

The mathematical model created from the experimental Modal Analysis may have a number of advantages over the pure analytical model based on spacial parameters:

- generally has many less independent variables than the number of measured DOFs
- expresses the true boundary conditions
- is easily verified

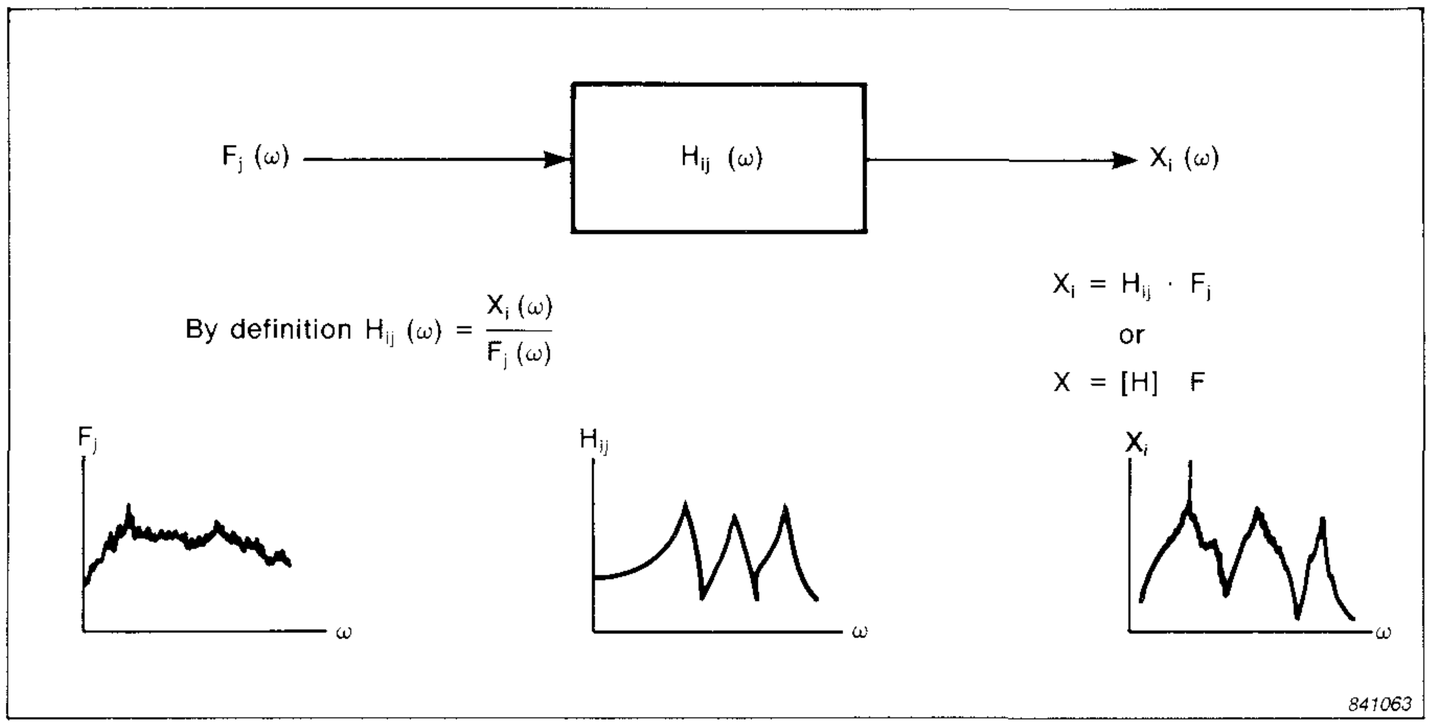

Fig. 11. The FRF and response model

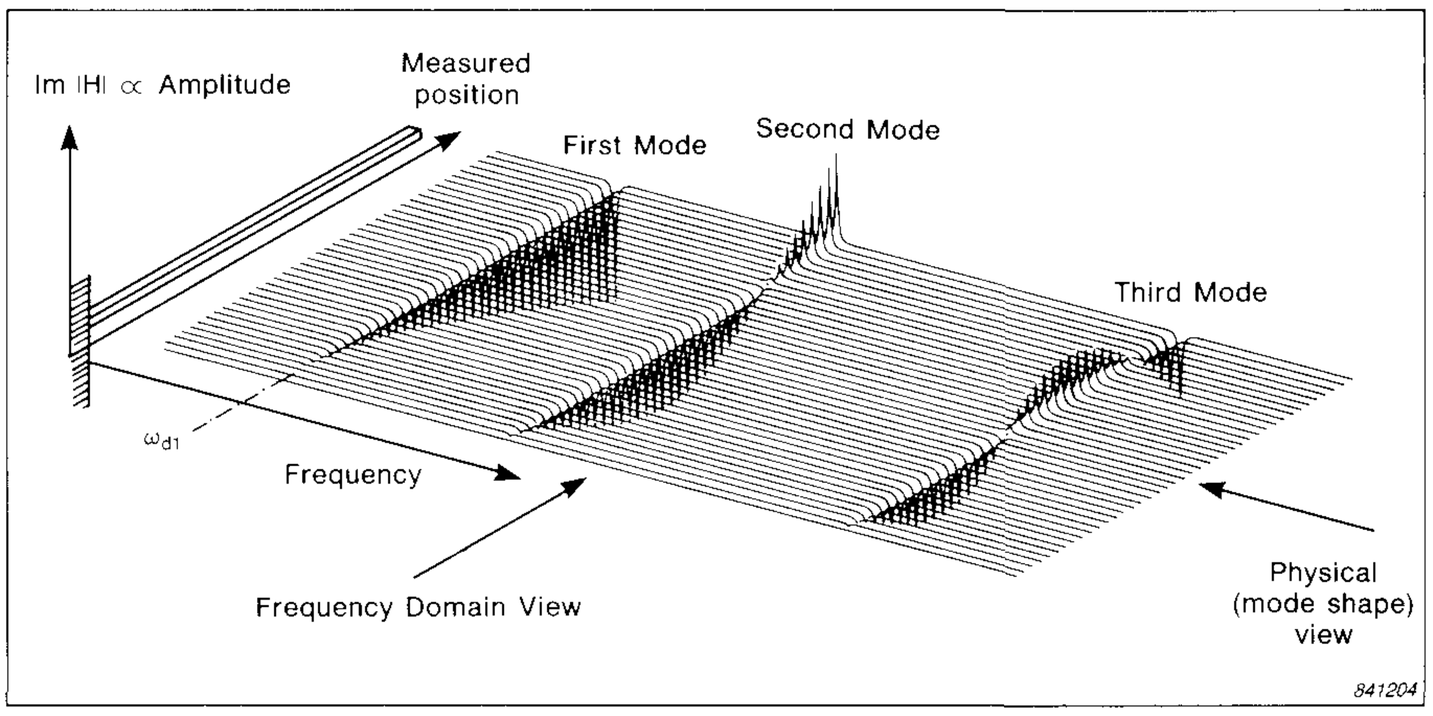

Fig. 12. Illustration of the mode shapes as derived from FRFs

FRF — Model

Analytically a general model for the compliance function is:

$$H_{ij}(\omega) = \sum_{r=1}^{m} \frac{R_{ijr}}{j\omega - p_r} + \frac{R_{ijr}^*}{j\omega - p_r^*} \quad [17]$$

where $R_{ijr} = k_r \cdot \phi_{ir} \cdot \phi_{jr}$ is the residue

$*$ indicates the complex conjugate

r is the mode index

i is the response index

j is the excitation index

$p = -\sigma + j\omega_d$ is the pole location representing natural frequency and damping

m is the number of modes

ϕ_{ir} is the scaled modal deflection in i for mode r

k is a scaling factor

For a system with only a single degree of freedom this model reduces to:

$$H(\omega) = \frac{R}{j\omega - p} + \frac{R^*}{j\omega - p^*} \quad [18]$$

These parametric forms of the FRF contain only measurable modal parameters.

All the modal parameters may be conveniently assembled in two matrices:

Modal Matrix:
$[\Phi] = [\{\phi\}_1 \ \{\phi\}_2 \ \ldots \ \{\phi\}_m]$

Spectral Matrix:
$[p] = [-\sigma_r + j\omega_{dr}]$

where σ is the decay rate, representing the damping

ω_d is the damped natural frequency

The dimensions of these matrices are for [Φ]: n × m and for [p]: m, where n is the number of DOFs and m is the number of modes. As the spectral matrix is diagonal, only m pole locations are contained.

Curve Fitting

Having measured an FRF, the models given in equations [13] and [14] can be used for estimating the modal parameters.

Each measured Frequency Response Function contains a very large amount of data. The B&K Dual Channel Signal Analyzers Types 2032 and 2034 give 801 complex numbers.

For each mode of vibration we need only identify two parameters: the pole location and the Residue.

This implies that we have a lot of redundant data available in the measurements. These data can be utilized by a curve fitting technique where, in a least squares sense, all observations are included to give the best estimate.

For a lightly coupled system, the single DOF model [18] can be used in a frequency band around each resonance, using between 10 and 200 values of H to estimate the two parameters R and p for each mode at a time.

If the structure exhibits FRFs with coupled modes, the modal parameters have to be estimated simultaneously using the multi-DOF model [17].

A high number of different estimation, or curve fitting techniques are available, but in general they are all based on the Least Squares principle. For parameter estimation see Refs. [2–4]

Dynamic Models

Dynamic mathematical models of mechanical structures are needed and extensively used in Structural Analysis. Simulations may be made:

- For predicting structural response from assumed external forces, with the purpose of evaluating the quality of the design.

- For predicting the dynamic effect of structural changes from mechanical improvements, adding pay load, coupling of substructures, etc.

The ultimative goal of Modal Analysis is to create a mathematical dynamic model representing the real structure.

In analytical analysis a time domain model may be formulated based on spacial parameters: mass, stiffness and damping matrices, see Fig. 13 a. The model consists of a set of simultaneous differential equations.

Transformation to the frequency or Laplace domain reduces the problem to a set of algebraic equations. The eigenvalues and eigenvector for the system matrix [B] (Fig. 13 b) represent the modal parameters.

In practice, Modal Analysis yields a frequency domain model directly (Fig. 13 c).

Using the modal matrix [Φ] a modal transformation is defined (Fig. 13 d):

$$\{x\} = [\Phi]\ \{q\} \qquad [19]$$

{x} represents the deflections in physical coordinates and {q} is the deflection in a system of modal coordinates. The number of DOFs in these generalized coordinates is reduced to the number of modes.

Using the modal transformation the equation of motion can be expressed in an uncoupled form (Fig. 13), based on modal parameters.

This means that a time domain as well as a frequency domain mathematical model can be constructed based on measured modal parameters.

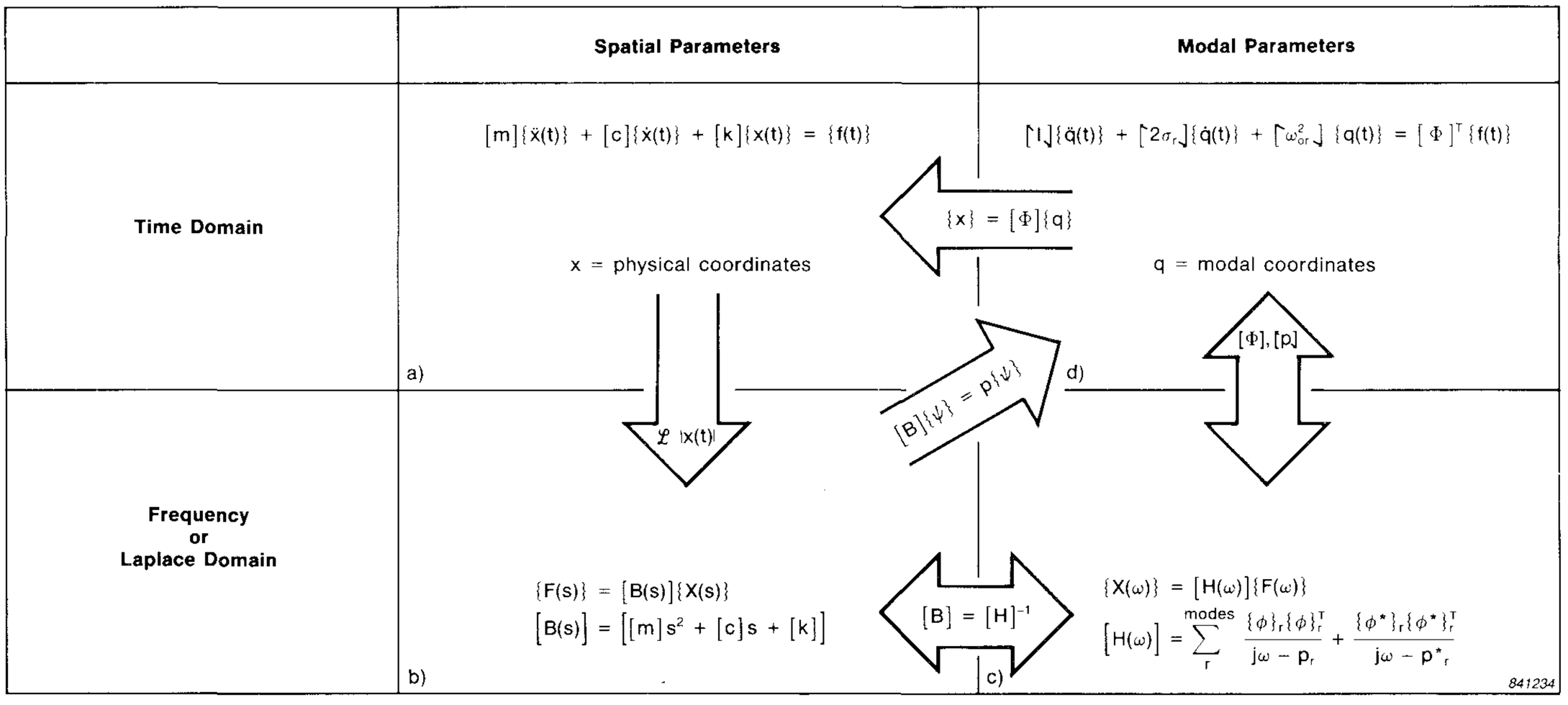

Fig. 13. Mathematical interrelation between Spacial Parameters and Modal Parameters for both the Time and Frequency domains

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