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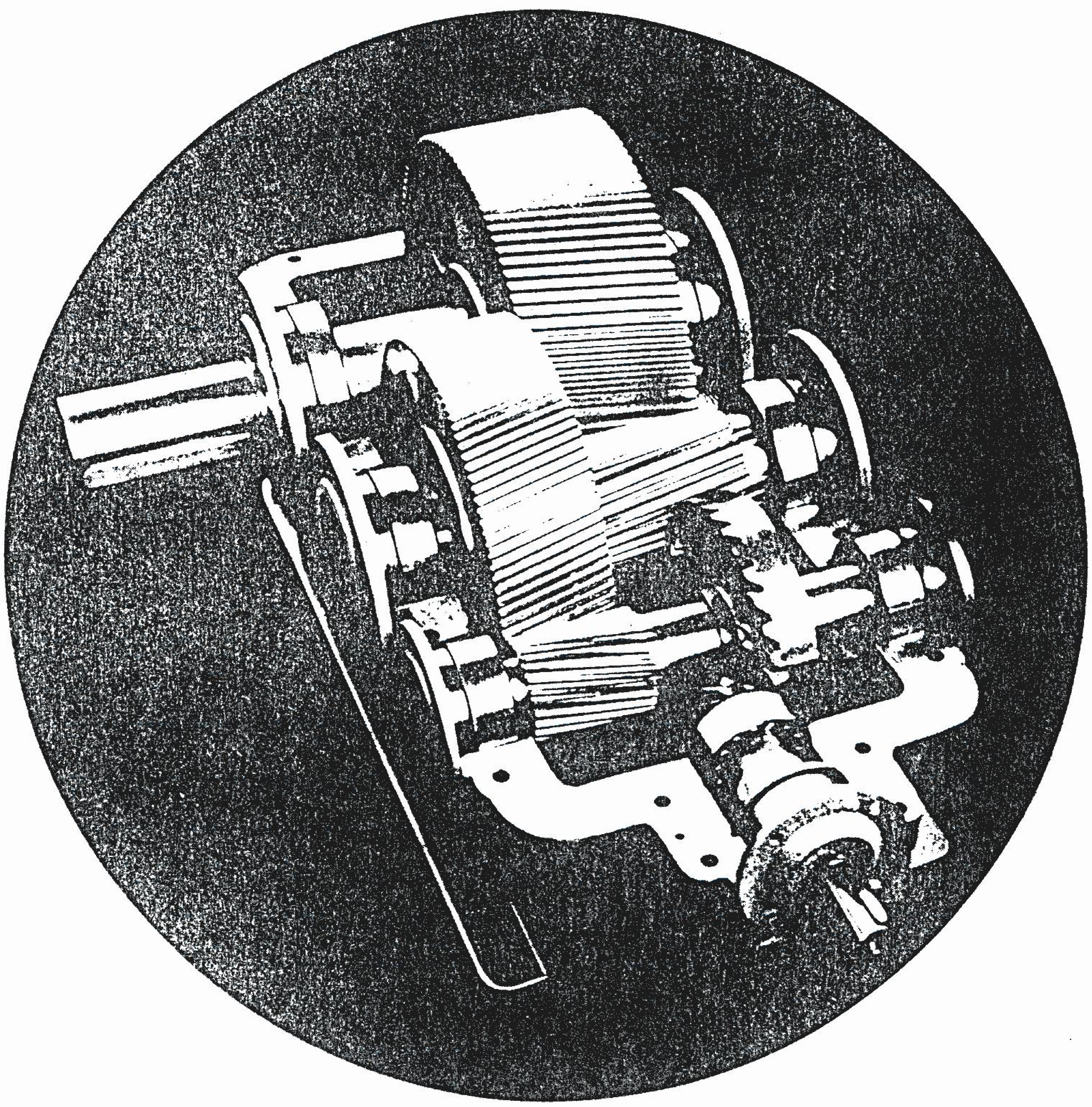
Cepstrum Analysis and Gearbox Fault Diagnosis



CEPSTRUM ANALYSIS

AND GEARBOX FAULT DIAGNOSIS

by

R.B. Randall B.Tech., B.A.

INTRODUCTION

Cepstrum Analysis is a tool for the detection of periodicity in a frequency spectrum, and seems so far to have been used mainly in speech analysis for voice pitch determination and related questions. (Refs. 1, 2) In that case the periodicity in the spectrum is given by the many harmonics of the fundamental voice frequency, but another form of periodicity which can also be detected by cepstrum analysis is the presence of sidebands spaced at equal intervals around one or a number of carrier frequencies.

The presence of such sidebands is of interest in the analysis of gearbox vibration signals, since a number of faults tend to cause modulation of the vibration pattern resulting from tooth meshing, and this modulation (either amplitude or frequency modulation) gives rise to sidebands in the frequency spectrum. The sidebands are grouped around the tooth-meshing frequency and its harmonics, spaced at multiples of the modulating frequencies (Refs. 3, 4, 5), and determination of these modulation frequencies can be very useful in diagnosis of the fault.

The cepstrum is defined (Refs. 6, 1) as the power spectrum of the logarithmic power spectrum (i.e. in dB amplitude form), and is thus related to the autocorrelation function, which can be obtained by inverse Fourier transformation of the power spectrum with linear ordinates. The advantages of using the cepstrum instead of the autocorrelation function in certain circumstances can be interpreted in two different ways. As regards sidebands, it means that by virtue of the logarithmic conversion more weight is given to low level components, and this is advantageous where it is primarily the existence of periodicity which is to be confirmed, and its frequency spacing accurately determined (see Fig. 1). In other applications such as speech analysis, the advantage is perhaps more that multiplicative relationships in the spectrum (e.g. by transfer functions) become additive on taking logarithms, and this additive relationship is maintained by the further Fourier transformation, thus eliminating the convolution (or "smearing") which would otherwise result (Ref. 1).

After a discussion of some basic concepts and definitions, the present article discusses the various ways in which the cepstrum can be obtained, and finally gives some results of the application to gearbox fault diagnosis. Much of the information is of course generally applicable, e.g. to other applications of cepstrum analysis and to other cases where it is desired to detect sideband growth.

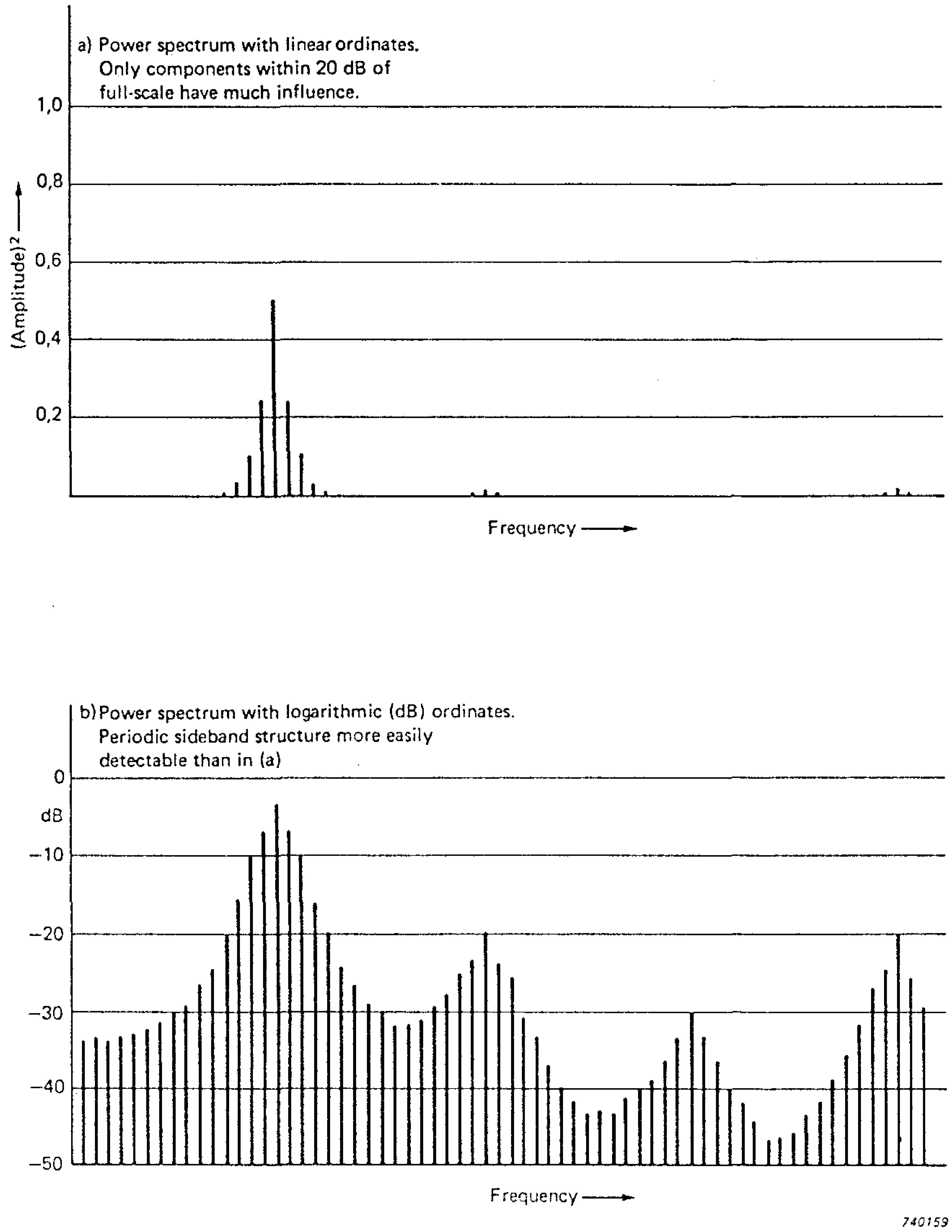


Fig. 1 Detection of periodicity in a spectrum

DEFINITIONS

The term "cepstrum" appears to have been first coined (from "spectrum") by Tukey et al. (Ref. 6) along with similarly derived terms such as "quefrency", "saphe", and "rahmonics" (from frequency, phase, harmonics).

The following definitions are used in the present discussion.

Cepstrum

This is normally defined as the power spectrum of the logarithm of the power spectrum. Since absolute calibration is of secondary importance (provided consistency is maintained) and since the logarithmic power spectrum would normally be expressed in dB, the unit of amplitude of the cepstrum is herein taken to be $(dB)^2$. On occasion, however, the term cepstrum may also be applied to the amplitude spectrum (square root of the power spectrum) and this will be distinguished by having the units dB.

Quefrency

This is the independent variable of the cepstrum and has the dimensions of time as in the case of the autocorrelation. The quefrency in seconds is the reciprocal of the frequency *spacing* in Hz in the original frequency spectrum, of a particular periodically repeating component. (Fig. 2) Just as the frequency in a normal spectrum says nothing about absolute time, but only about repeated time intervals (the periodic time), the quefrency only gives information about frequency spacings and not about absolute frequency.

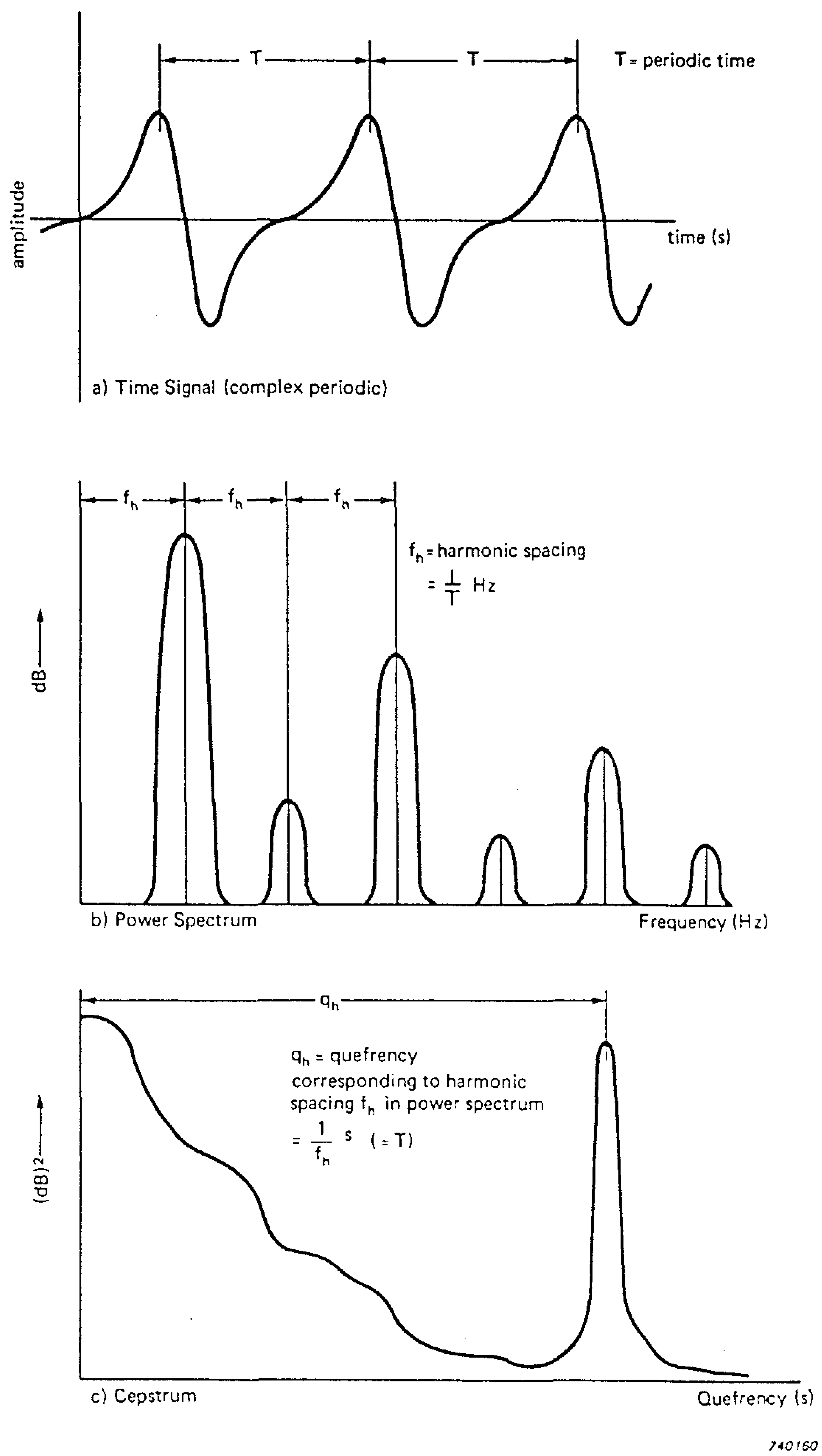


Fig. 2 Time signal, log spectrum and cepstrum for a periodic signal

METHODS OF OBTAINING THE CEPSTRUM

Since the originator of the technique, Tukey, was also one of the originators of the Fast Fourier Transform (FFT) algorithm, it is natural that up until now digital techniques have been used almost exclusively for cepstrum determination. This also represents a very efficient and rapid method since the same algorithm is used for obtaining both the original spectrum and the cepstrum, and operation in Real Time is feasible. Several other possibilities are described here, however, ranging from completely analogue set-ups to arrangements where the spectrum analysis is carried out by analogue methods, though with spectrum output in digital form for digital computation of the cepstrum. These methods will often be less expensive in instrument cost than digital analysis, and many people already possess the necessary instruments.

Calibration techniques are slightly different for each method and are discussed as appropriate.

Method 1

A typical set-up is shown in Fig. 3. The spectrum is taken from the "log DC" output of the 2010 Analyzer and recorded on a 7502 Digital Event Recorder. The longest recording time available (using the internal clock of the 7502) is 100s, but in many cases it will be possible to carry out the complete spectrum analysis in this time. This applies for example to the high-speed gearboxes discussed later where the tooth-meshing frequency is as high as 3–10 kHz. (See Ref. 7 for selection of optimum analysis speed.) If analysis time is longer it will be necessary to use an external clock. Where the original signal is recorded on an endless loop (tape loop, or another 7502) then a once-per-revolution pulse can often be used as this external clock. The Analyzer type 2010 can be driven either mechanically or electrically by the level recorder (which is required for later recording of the cepstrum) or electrically from an external ramp generator.

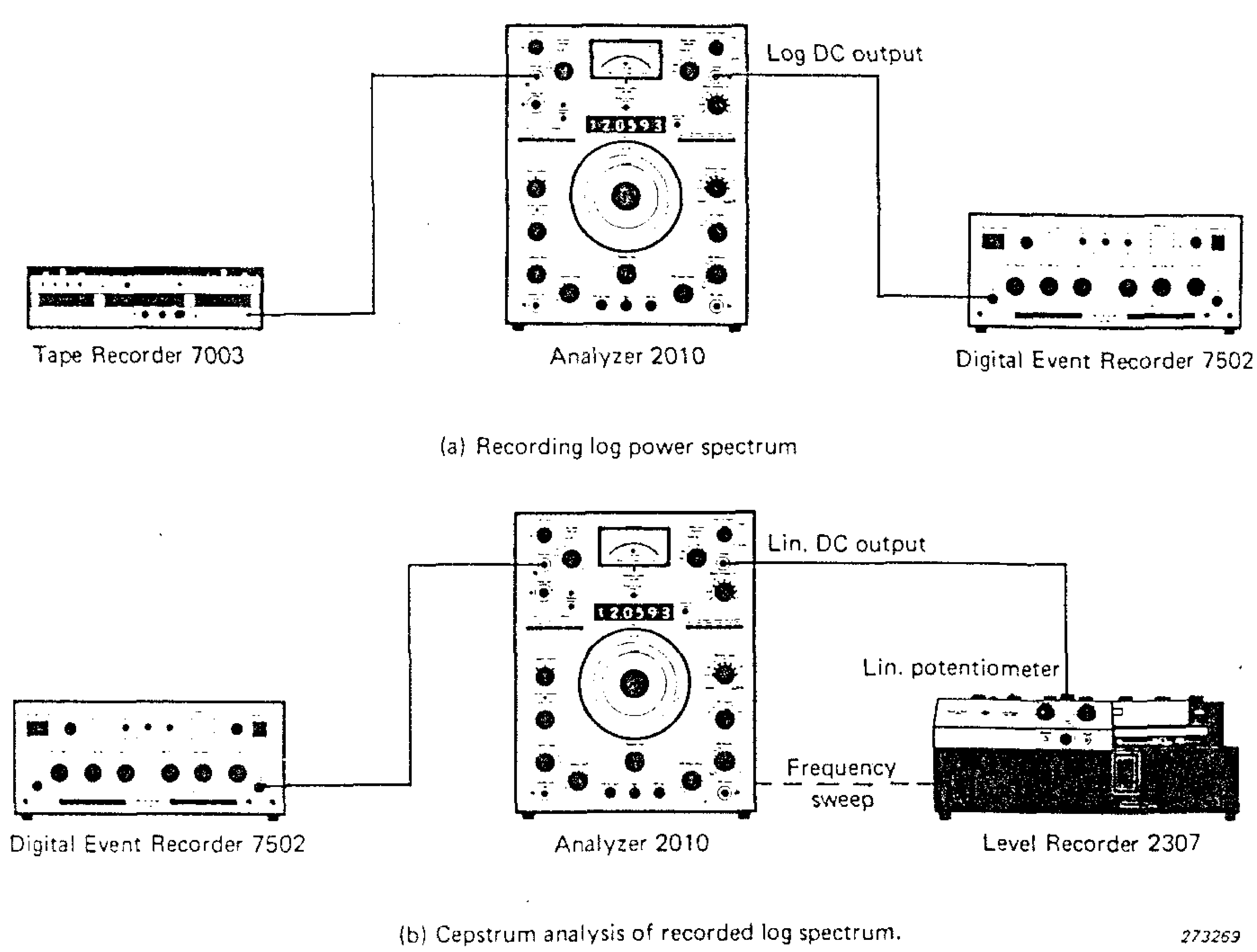


Fig. 3 Instrument set-ups for Cepstrum Analysis (Method 1)

On playback of the logarithmic spectrum, the actual cepstrum analysis is performed like a normal frequency analysis, with the exception that a linear potentiometer is used in the Level Recorder. Because of the wide frequency range of the 2010, the 7502 could virtually always be played back at the highest speed of 500 kS/s (unless an intermediate resolution bandwidth is desired). The 20 dB linear potentiometer (Type ZR 0002) is recommended as giving the best range, and it may be necessary to experiment a little to find the best attenuator settings since the low-"quefrency" components will generally not be of interest and can be allowed to exceed full-scale. Fig. 4 shows a typical cepstrum obtained in this way from the sound power spectrum of an electric drill (primarily gear noise).

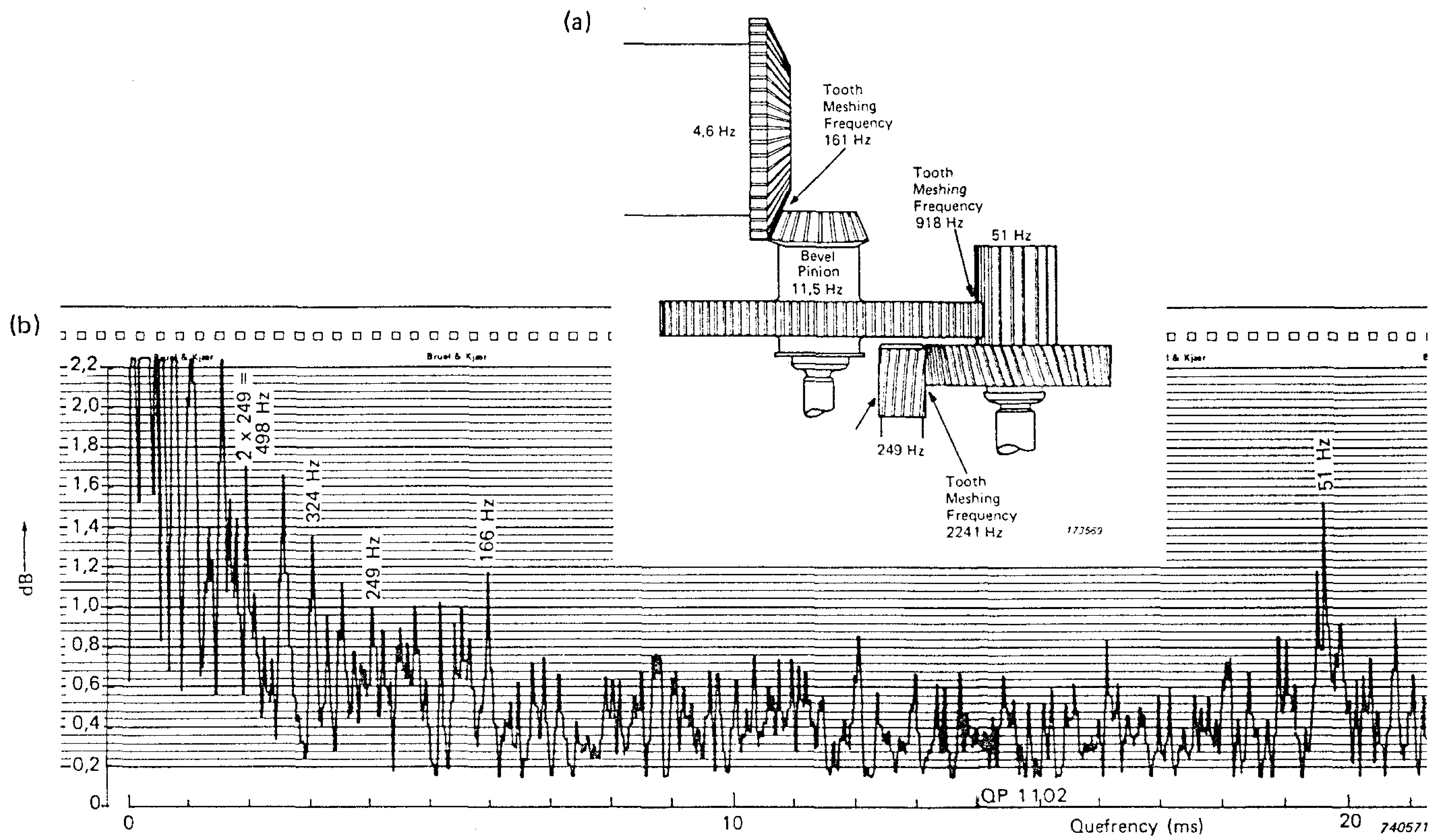


Fig. 4 (a) Gear arrangement and frequencies
(b) Cepstrum obtained by Method 1

Since the analyzer output voltage is proportional to the RMS rather than the mean square value, it is the amplitude spectrum which is recorded, with the units of dB from the original analysis. Amplitudes can be calibrated from the fact that the 50 dB dynamic range of the "log DC" output of the 2010 is equivalent to the voltage range 0–4.5 V.

For calibration of the quefrency scale, the following procedure can be used:

If spectrum analysis sweep speed = S_s Hz/s

and if the recording sampling rate of the 7502 = R_S samples/s

then the frequency spacing of the samples = $\frac{S_s}{R_s}$ Hz

This can then be related to the frequency range on playback as follows:

$$\text{Sampling "quefrency"} = \frac{1}{\text{frequency spacing}} = \frac{R_s}{S_s} \quad \text{seconds}$$

which is equivalent to the 7502 sample rate on playback P_s (e.g. 500 kHz)

Thus, an arbitrary final analysis frequency f_a represents the quefrency

$$q_a = \frac{f_a}{P_s} \cdot \frac{R_s}{S_s} \text{ (s)} \qquad (1)$$

and this in turn is equivalent to the frequency spacing (e.g. sideband spacing or modulating frequency).

$$\frac{P_s}{R_s} \cdot \frac{S_s}{f_a} \text{ (Hz)} \qquad (2)$$

Method 2

This is an extension of Method 1 for the case where the original analysis time would otherwise be excessive, or where (as with speech analysis) it is necessary to select out a particular sample from a longer, perhaps non-stationary, record. A typical set-up for obtaining the logarithmic spectrum is shown in Fig. 5 and this is the only difference from Method 1. The first 7502 is used for recording a sample of the original time signal, and the second for recording the log spectrum. The analysis can thus be carried out at high speed, and possibly using the Gauss Impulse Multiplier Type 5623 to select out a particular sample. (See Ref. 8 for details.)

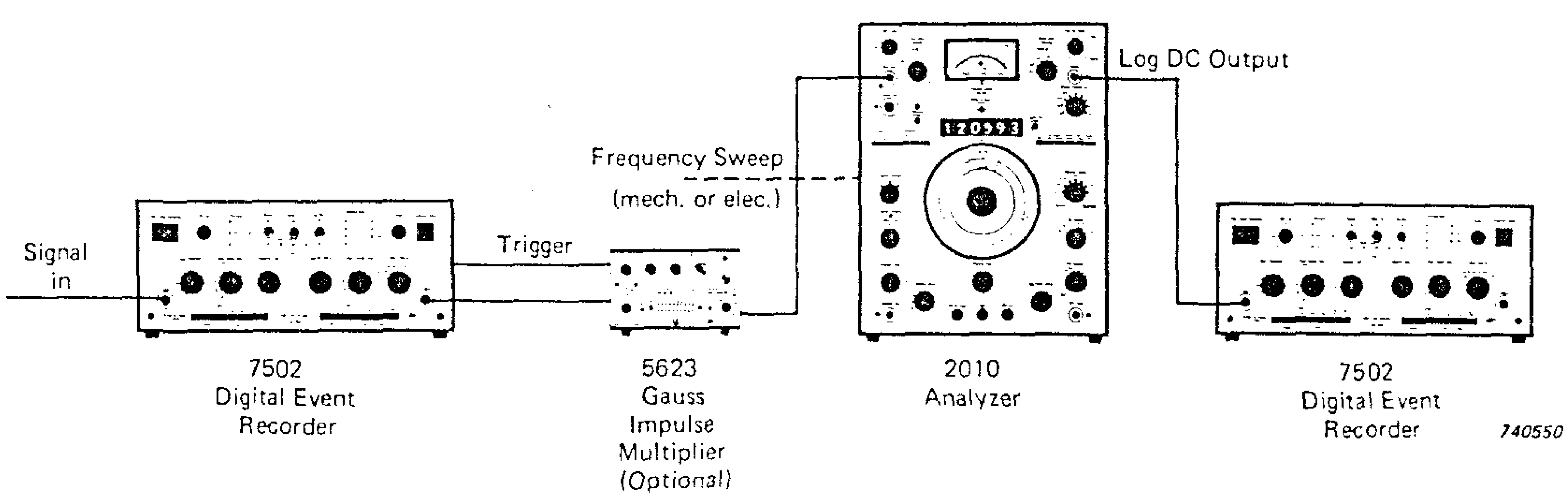


Fig. 5 Instrument set-up for Method 2

In cases where the Gauss impulse multiplier is used, the averaging time will be several times the 7502 memory circulation time (Ref. 7) and it would be possible to use either the internal clock of the second 7502 (in which case the internal antialiasing filters can be used), or the "Sync. Trigger" pulse of the first as an external clock, giving one sample per record circulation.

Where the signal is stationary and the first 7502 is used purely to speed up the analysis there is a further possibility, since the averaging time can be made equal to or less than the memory circulation time and the Sync. Trigger pulse of the first 7502 can be used as an external clock for the second.

Since the sample is then taken before it can respond to passage of the signal discontinuity (which is coincident with the Sync. Trigger pulse) the effect of this can be greatly reduced to the point where the Gauss Impulse Multiplier may not give any advantage. The background for this is described fully in Ref. 9, where the identical question of spectrum sampling using the Digital Encoder Type 4421 is discussed.

Method 3

It is of course possible to replace one or both Digital Event Recorders by Tape Recorders Type 7003 (generally with tape loop adaptor mounted) but analysis time would be considerably longer since the maximum speedup ratio is 10. Equations (1) and (2) can still be used, with the ratio $\frac{P_s}{R_s}$ replaced by the speedup ratio between recording and playback of the log spectrum. (Normally 10 for the 7003.)

Method 4

The Real Time Analyzer Type 3348 can be used directly as a Cepstrum analyzer provided that the optional X–Y Recorder Control Type ZH 0107 is mounted.

The spectrum is first derived in the normal manner and stored in for example Store A. The spectrum, logarithmically converted, is then read out in analogue form from the X–Y Recorder output at the maximum rate (36 s) and fed into the analyzer input with frequency range set at 10 Hz (40 s record length). The read-out can be followed visually on the screen, and there is thus 4 s in which to "hold" the spectrum in the analyzer memory without loss of information. The cepstrum can then be read into Store B, while the spectrum is retained in Store A, meaning that either can be viewed or read out to a recorder at will.

Amplitude Calibration is determined by the fact that the 50 dB dynamic range at the Recorder Output represents the voltage range 0–5 V. (100 mV/dB.)

Quefrency Calibration is as follows:

Frequency range setting for original analysis = f_{R1} Hz

This is equivalent to $\frac{36}{40}$ or 90% of the total record length for the second analysis. The frequency spacing of each line in the final analysis = $\frac{1}{\text{record length}} \equiv \frac{0.9}{f_{R1}}$ seconds quefrency and the full scale quefrency is 400 times this or

$$\frac{360}{f_{R1}} \text{ seconds} \tag{3}$$

independent of the scale range used for the second analysis. (Having captured the logarithmic spectrum in the 10 Hz range, it can be an advantage to average its spectrum, using a higher frequency range for high speed, and thus averaging out the internal noise of the analyzer.)

Fig. 6 shows a typical spectrum and cepstrum obtained in this way for the vowel sound "aaaa".

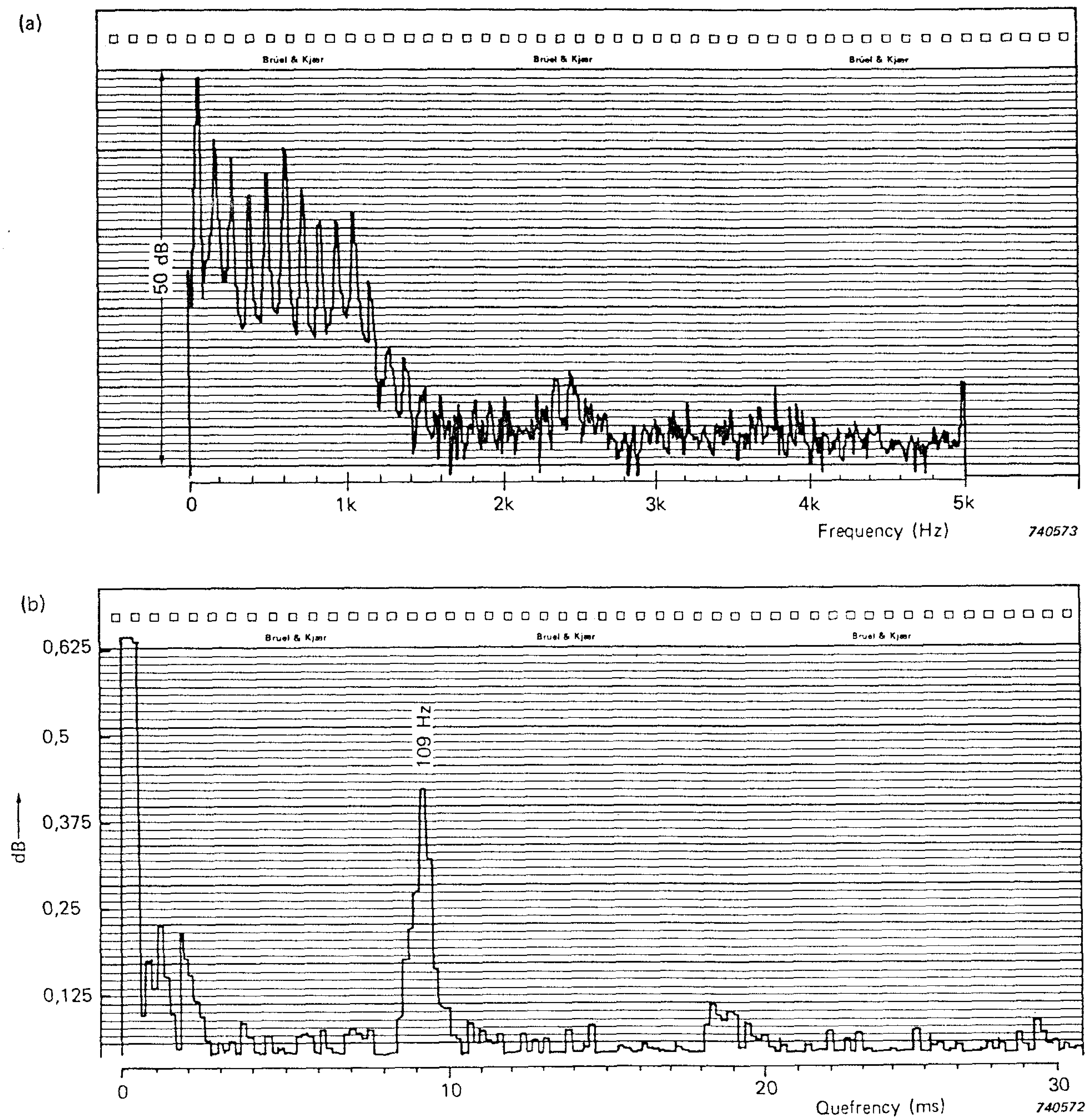


Fig. 6 (a) Spectrum of vowel "a a a a"
(b) Cepstrum of vowel "a a a a"

Method 5

If the spectrum is obtained in ASCII-coded BCD form on punched-paper tape, it can be used as input data to many computer systems, including timesharing systems, for which the normal operating languages are easily-learned high-level languages such as Fortran and Basic. Data in this form can be obtained from the set-up in Fig. 7 (Ref. 9) or from the 3348.

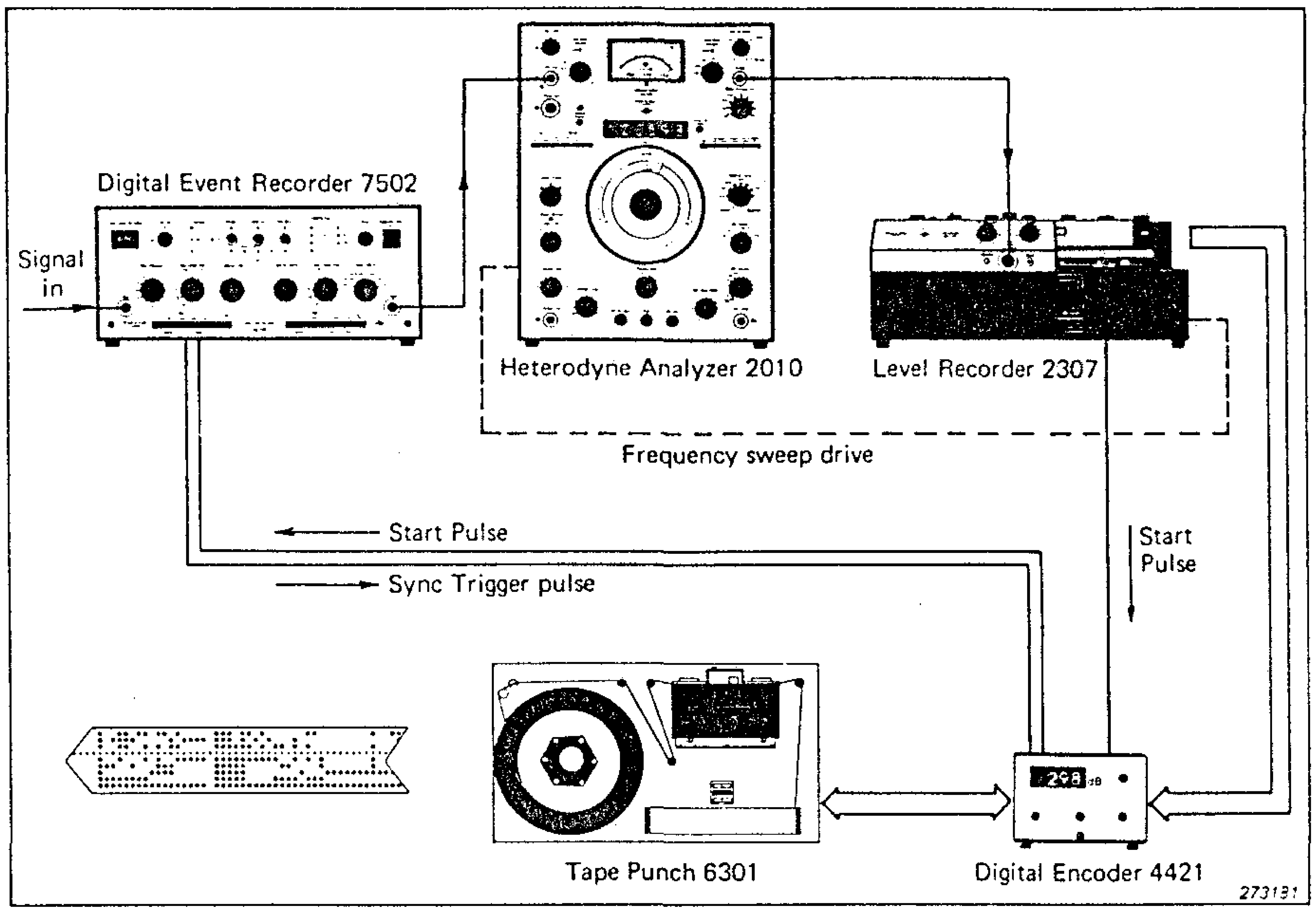


Fig. 7 Set-up for high speed analysis with digital output

In cases where the analysis can in any case be carried out fairly rapidly (eg. high speed gearboxes) then it would be possible to eliminate the 7502 from the set-up in Fig. 7, and allow the digital encoder to be triggered by the "frequency shift" pulses from the 2010, since constant bandwidth and linear frequency scale are used. Frequency spacings would of course be limited to powers of 10 Hz.

It is then quite feasible to calculate the cepstrum using a Fast Fourier Transform (FFT) program written in the high-level language.

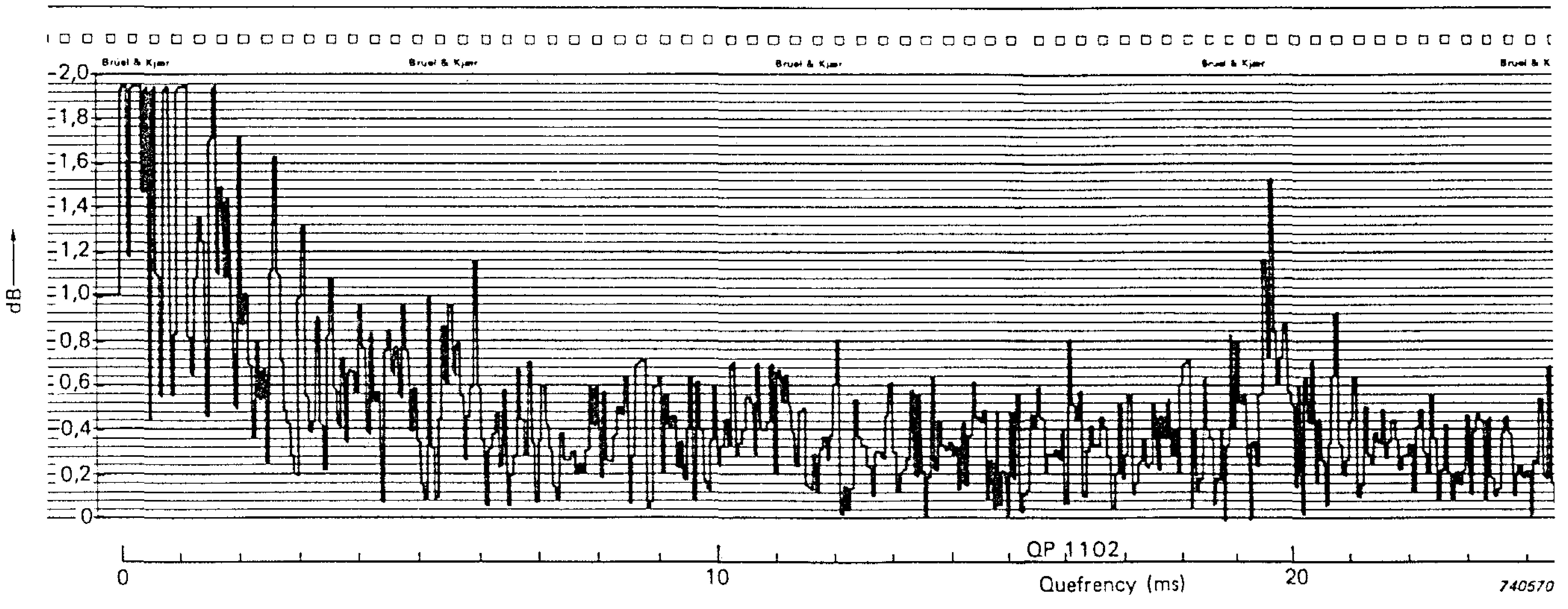


Fig. 8 Cepstrum obtained by Method 5 for same spectrum as Fig. 4

Fig. 8 shows the result of such a calculation on the same signal as for Fig. 4, and it can be seen that they are very similar. Note that it is possible with the digital method to express the result either as amplitude spectrum (units dB) or power spectrum (units $(dB)^2$) as desired; also that the vertical scale runs from zero whereas in Fig. 4 it starts at approx. 9% of full-scale (with the 10 – 110 mV potentiometer). The calculations were performed on a 16 K Varian computer with auxiliary disk unit (MOS System). The FFT program used (NLOGN) was obtained from Ref. 10 and is reproduced in Appendix A together with the subroutine HALF, based on the method given in Ref. 11, which uses NLOGN to perform the forward transform more efficiently when the input data are real. Both subroutines are written in Fortran IV. Appendix A also gives a guide to the use of the FFT algorithm.

Presentation of the results is easiest when a digital plotting facility is available. Figure 8 was in fact obtained by a devious process involving conversion of the results from BCD form into 8-bit binary which could then be read into a digital Event Recorder Type 7502 for plotting on a Level Recorder, but this is not generally practicable.

The question might well be asked "Why not do the original frequency analysis using the same procedure", but in fact it is hardly practicable. For one thing it is normally necessary to average over several analyses (at least 10) to obtain a reasonably reliable result, and moreover each of the spectrum analyses would have to be at least double the cepstrum analysis in length (Appendix A). Overall this would mean reading in more than 20 times as much data, which is not practicable with paper tape. The analysis time is also relatively long with the program written in a high-level language. (Typically 1 min.)

Amplitude Calibration is no problem in consideration of the fact that the BCD output of both the Digital Encoder Type 4421 and the Time Compression Analyzer Type 3348 is the same, viz. 0.1 dB per unit (e.g. 473 = 47.3 dB).

Quefrency Calibration is as follows:

Frequency spacing between samples = f_s Hz

$$\therefore \text{ Sampling "quefrency"} = \frac{1}{f_s} \text{ s}$$

The subroutine HALF gives an output array varying linearly in quefrency from zero (first value) up to the "Nyquist quefrency" (final value) and the Nyquist quefrency is one-half the sampling quefrency or

$$\frac{1}{2 f_s} \tag{4}$$

Method 6

This is a fully digital method consisting of a 7504 computer in combination with a 7502 Digital Event Recorder as shown in Fig. 9. A special interface is required between these two instruments, but because of the rapid data transfer between them, it becomes quite efficient to also perform the original spectrum analysis by FFT methods. With the program written in DAS assembler language, the total calculation time is a matter of seconds. The minimum computer size which can be used is 8 K, and then any other programming would have to be in assembler language. Maximum transform size is 4096 data points. Some details are given in Ref. 12.

A possible alternative is to use Varian's standard FFT subroutine package (No. A–983) which although written in machine language is callable from a BASIC program. Minimum computer size is, however, 16 K, and the maximum transform size is 2048 data points.

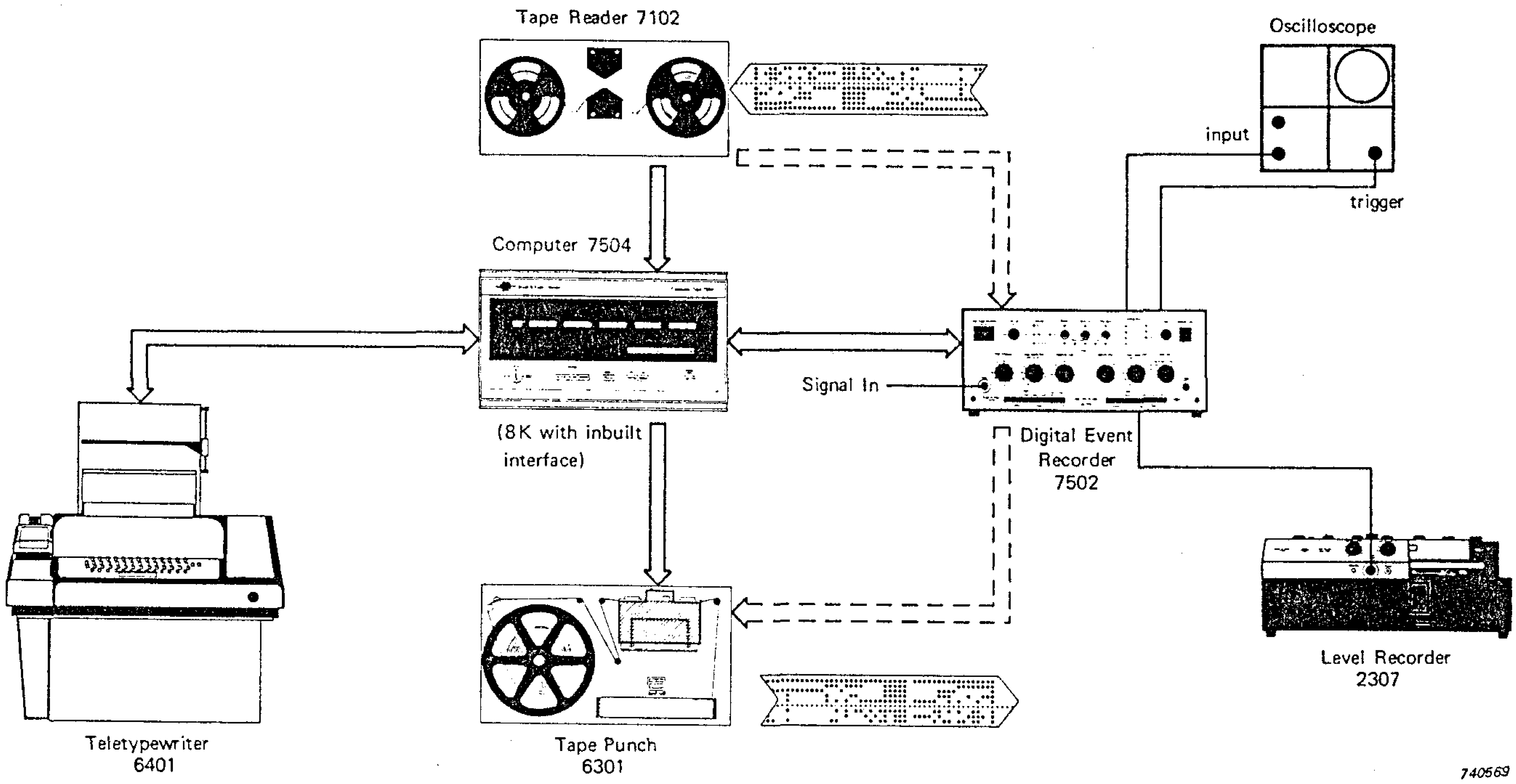


Fig. 9 Instrument set-up for FFT analysis

Examples

1. The first example is from a high speed gearbox mounted between a gas turbine and an alternator with shaft speeds of 85 Hz and 50 Hz respectively. Vibration recordings were made just prior to and shortly after a maintenance shutdown during which the gearbox internals were replaced. The reason for this was that during an inspection 6 months earlier, fretting corrosion had been discovered between the wheels and the shaft, but it had been possible to run in the intervening period while the new parts were being made. The first recording was made of acceleration, but an analysis of this showed that it was preferable to record velocity and so this was done on the second occasion.

Fig. 10 shows 3% bandwidth analyses of the two signals compared as velocity and it is interesting that although a considerable change is evident at the shaft speeds, due to realignment, the change at the tooth meshing frequency and its harmonics is not marked in this representation. Fig. 11 shows constant bandwidth spectra of the same two signals on a linear frequency scale (this time as recorded, since the overall slope of the spectrum is of secondary importance). The difference now shows up clearly in the large number of sidebands around the tooth meshing component and its harmonics in the signal taken "before repair", whereas the three harmonics stand out clearly "after repair". Fig. 12 shows cepstra corresponding to the spectra of Fig. 11 (using Method 5) and this confirms that the major modulating frequency is 85 Hz, and that it is much more marked "before repair".

It is worth noting, however, that other cepstrum analyses under different conditions have not given the same clear result, and this is thought to be due to the difference in noise levels in the two original recordings, since one was recorded as acceleration and the other as velocity. At least until the influence of such factors is completely clarified, it would seem to be wise to only compare signals processed under identical conditions.

2. The second example illustrates some of the measures which it is desirable to take. Both spectra and cepstra have been obtained by Method 6, from vibration signals recorded from a large slow speed gearbox driving a cement mill. The signals taken "before repair" show the result of many years' operation. At this time it was found necessary to replace a bearing, and the machine was then started up in the reverse direction, which could be expected to give the same effect as a new gearbox.

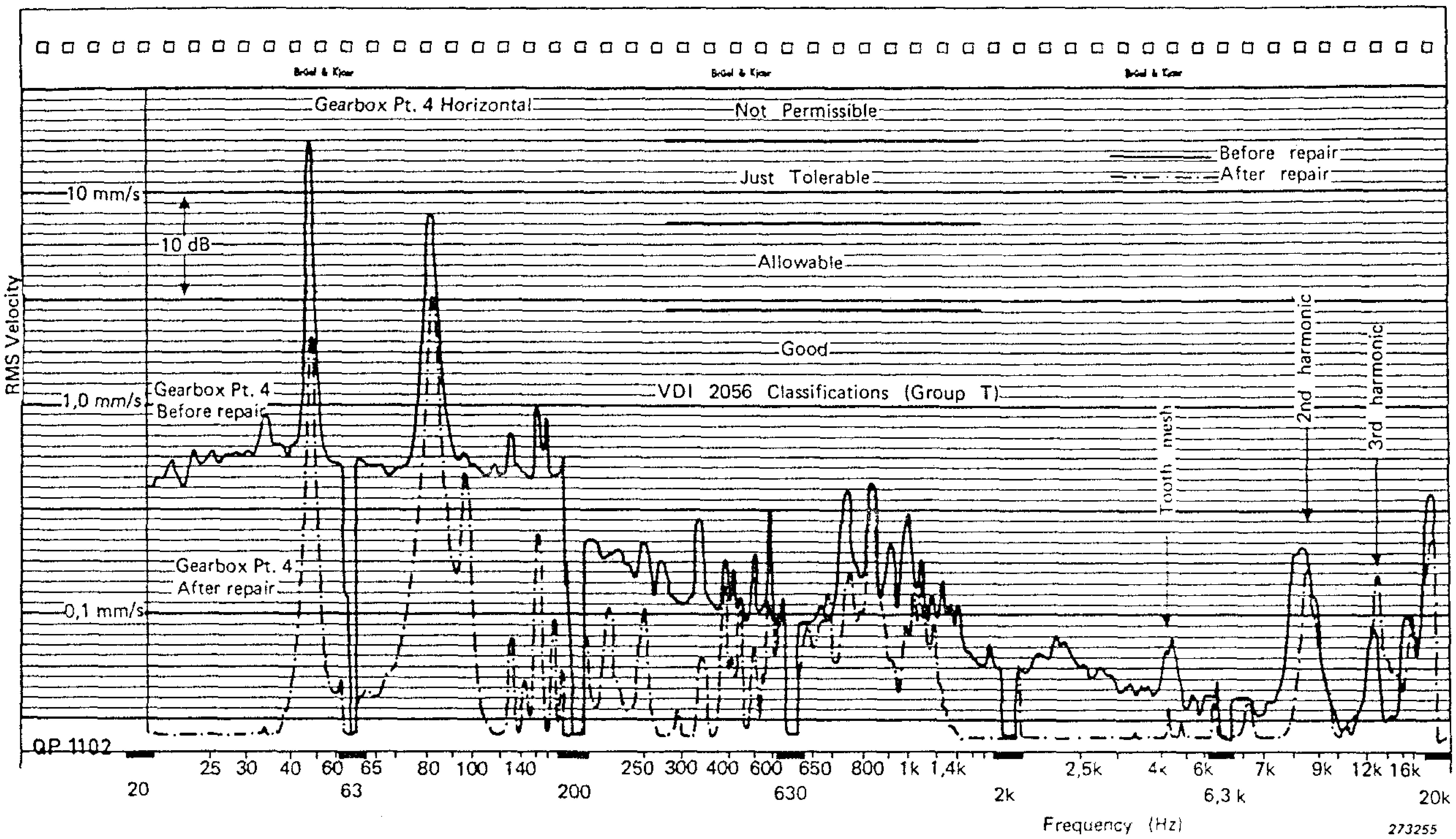


Fig. 10 High-speed Gearbox Vibration Spectra (3% bandwidth)

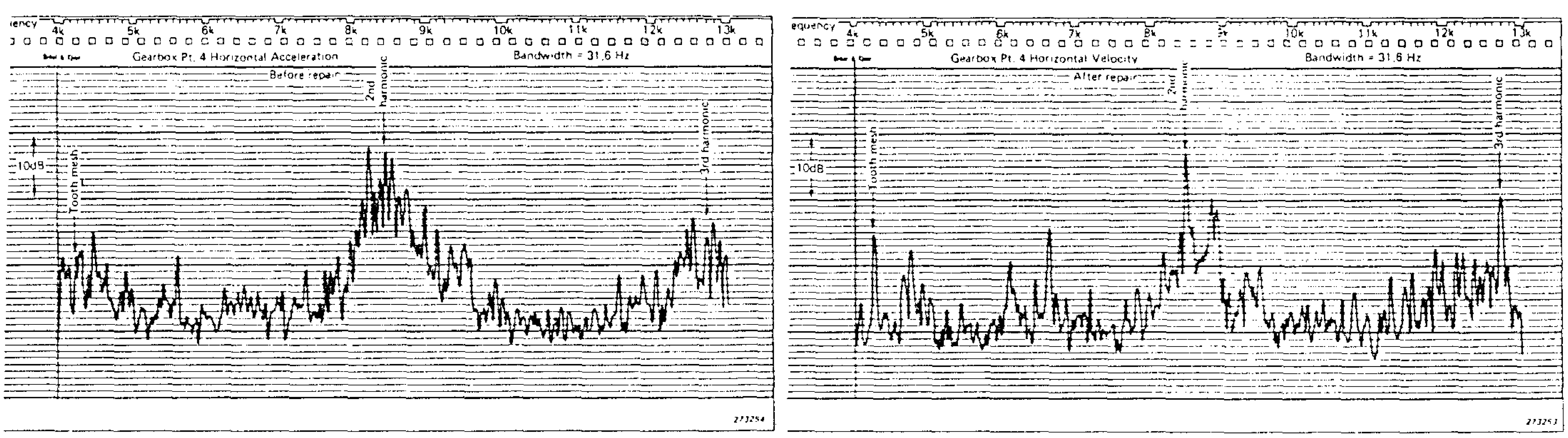


Fig. 11 High-speed Gearbox Vibration Spectra (constant bandwidth)

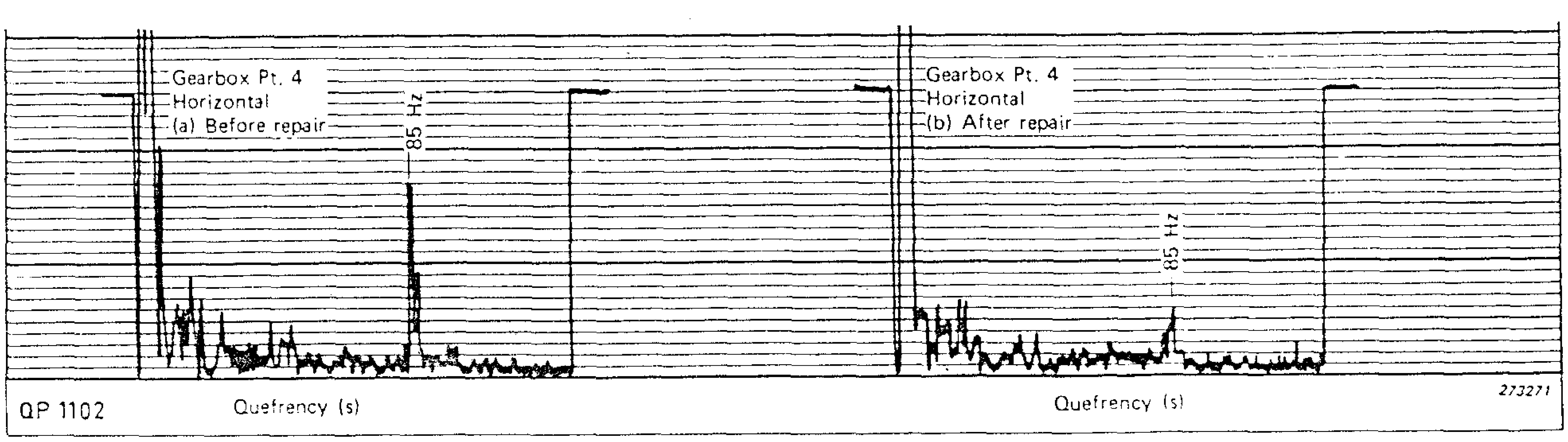


Fig. 12 Cepstra corresponding to Fig. 11

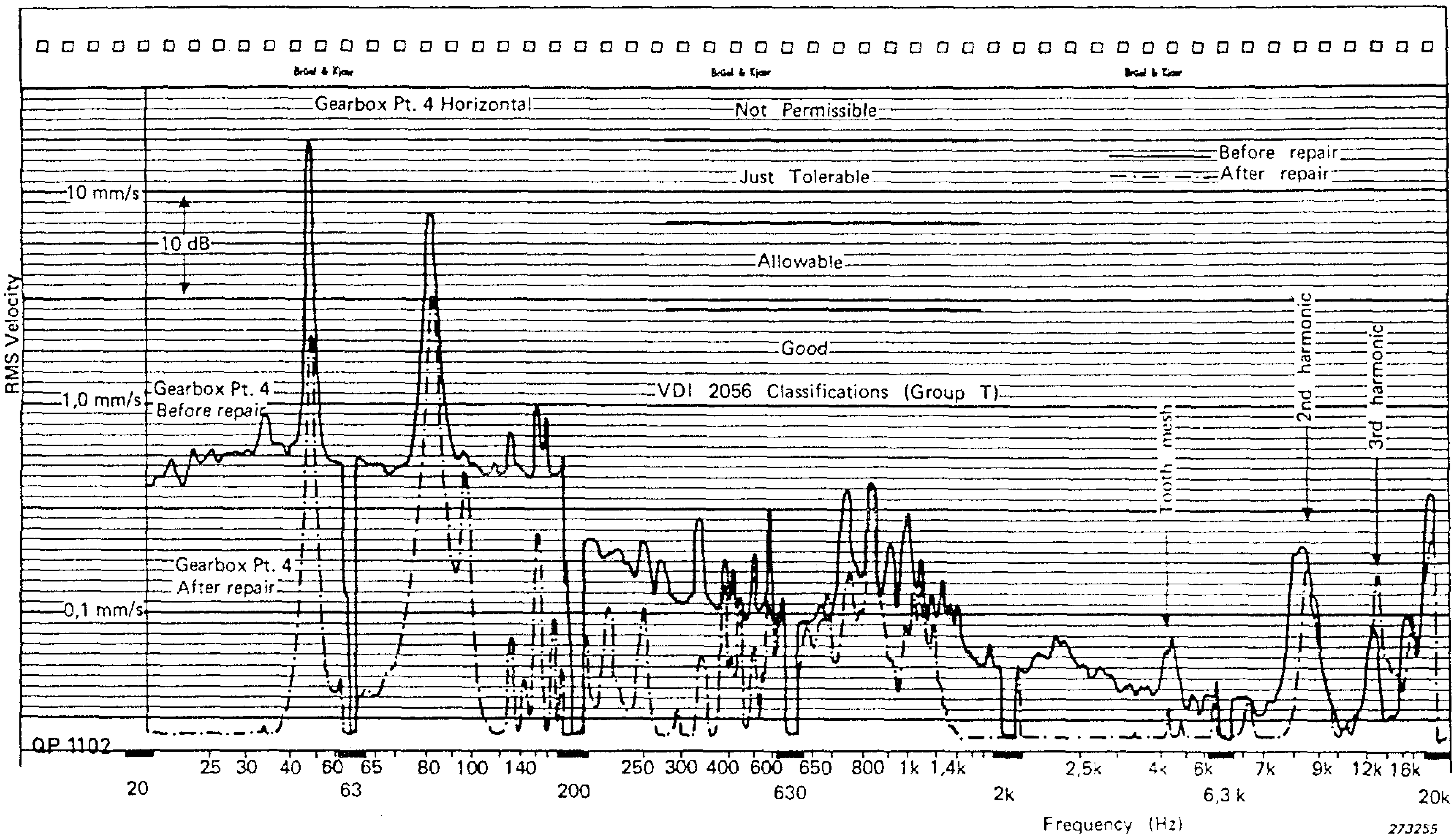


Fig. 10 High-speed Gearbox Vibration Spectra (3% bandwidth)

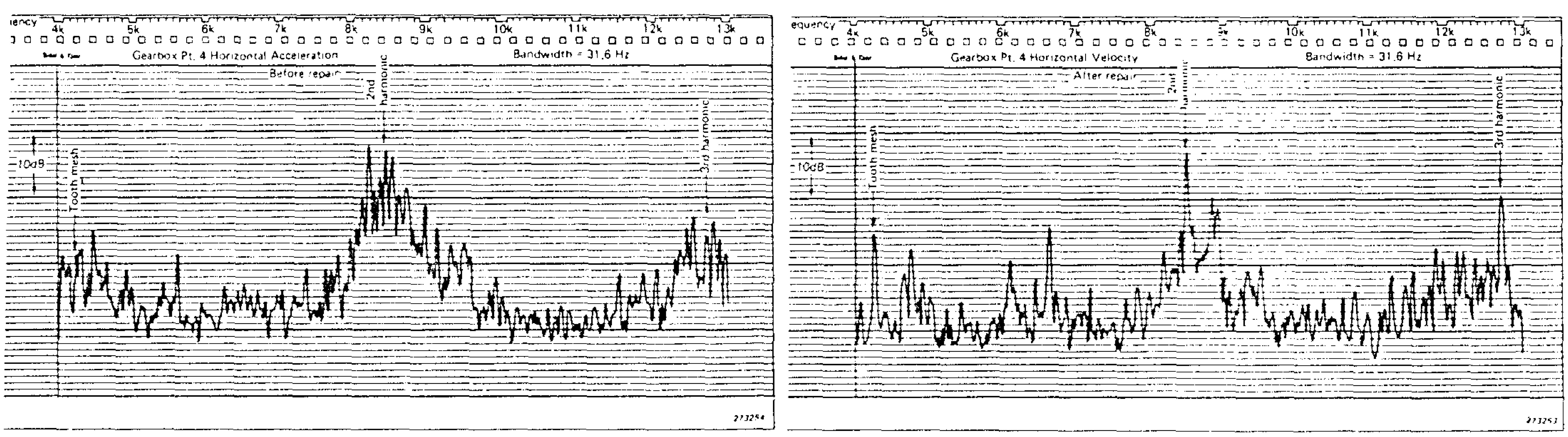


Fig. 11 High-speed Gearbox Vibration Spectra (constant bandwidth)

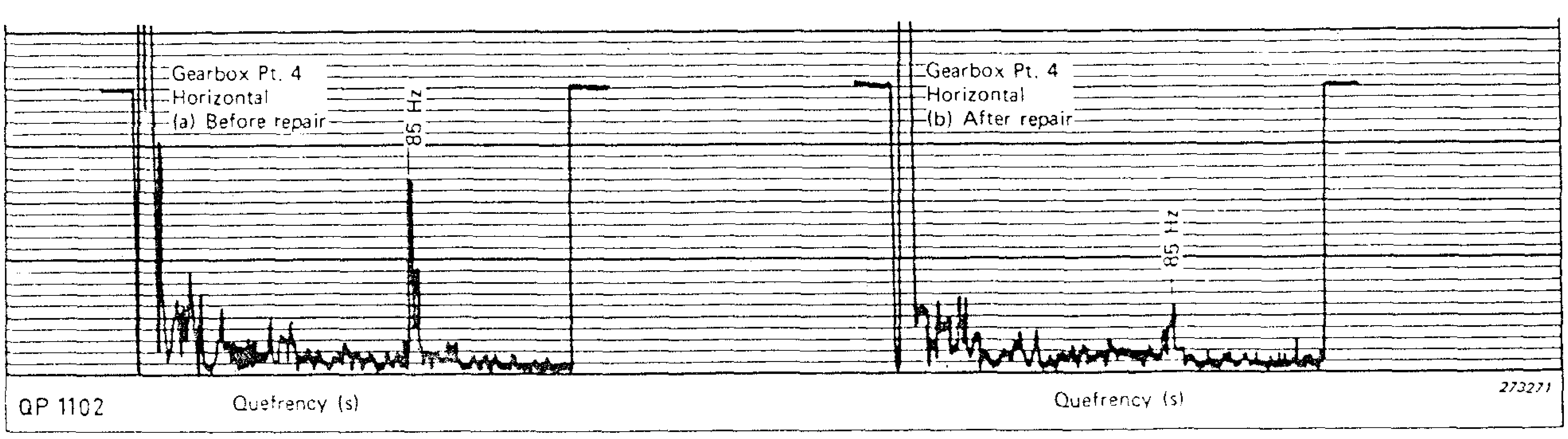
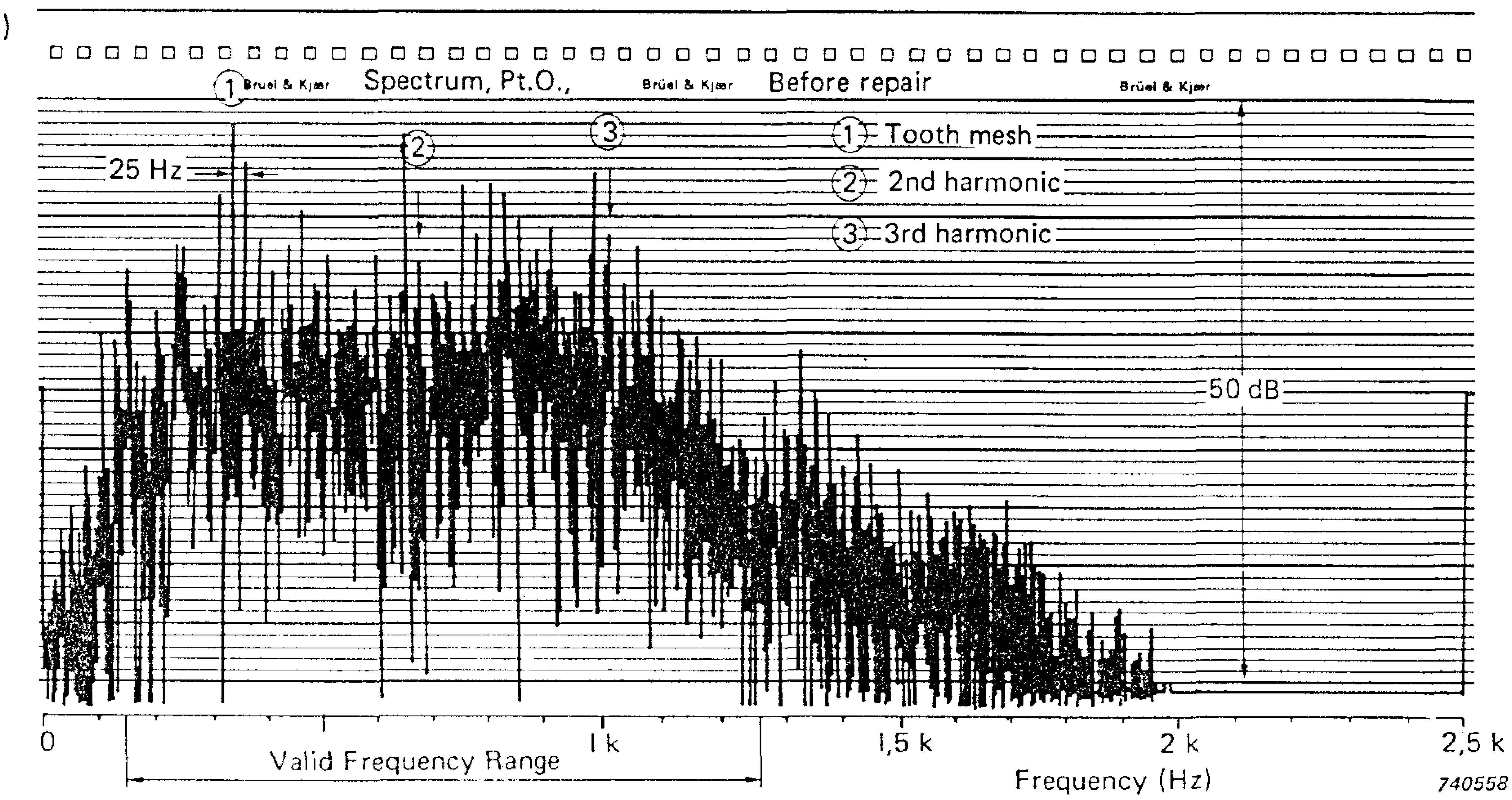


Fig. 12 Cepstra corresponding to Fig. 11

Figs. 13(a) and (b) show the spectra "before repair" and "after repair" respectively. They result from 4 K FFT transforms giving 2 K frequency points up to the Nyquist frequency and thus 1 K points up to the lowpass filter cutoff of the 7502. It has been arranged that the first three harmonics of the high speed tooth meshing frequency lie below this cutoff frequency. Furthermore, a highpass filter has been applied to the signal before recording on the 7502, with cutoff frequency at about a half of the tooth meshing frequency in order to eliminate the possible effect of low harmonics of the shaft speeds. The resulting valid frequency range which is illustrated thus includes frequencies from approx. $^1/_2$ to $3^1/_2$ times the tooth meshing frequency being investigated, and eliminates extraneous effects as much as possible.

(a)



(b)

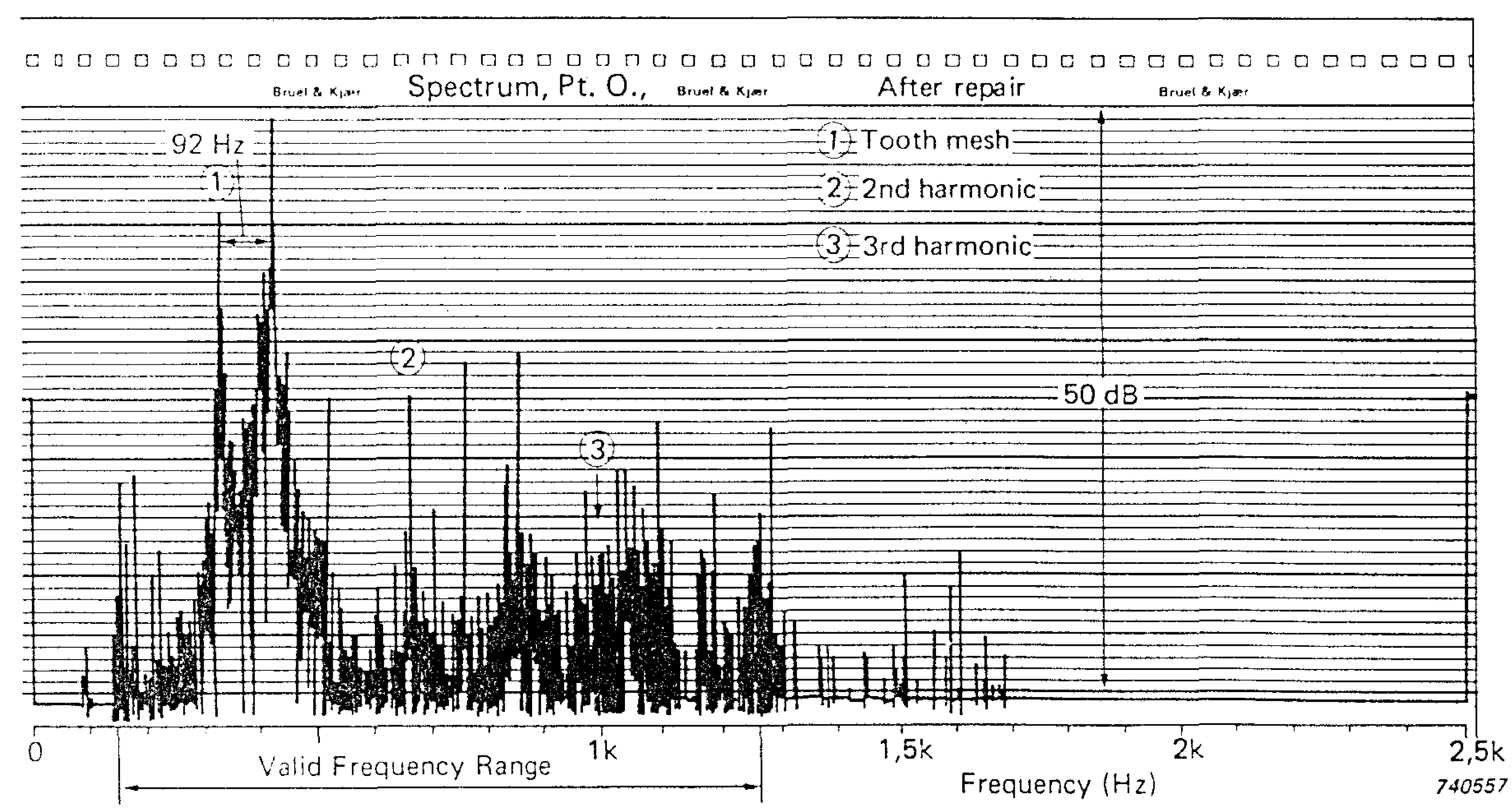


Fig. 13 Low-speed Gearbox Vibration Spectra
(a) Before Repair
(b) After Repair

There was an appreciable difference in the absolute levels in this case, but since this would only influence the DC component in the cepstrum, both spectra have been normalised to the peak component with a dynamic range of approximately 50 dB, and the spectra as illustrated have been used as input data to the second transform.

Figs. 14 (a) and (b) show the two corresponding cepstra. (Note the difference in amplitude scaling). What appears to be the dominant spacing in the spectrum "after repair", viz. 92 Hz (Fig. 13 (b)) which is in fact a major component in the cepstrum (Fig. 14 (b)) is thought to be a "ghost" component resulting from the gear cutting machine (Ref. 5), and could be representative of the normal state of affairs. It is present at about the same level in the cepstrum "before repair" (Fig. 14 (a)) though not then at all obvious in the spectrum (Fig. 13 (a)). The other major component in the cepstrum "after repair" (Fig. 14 (b)) is at 24 Hz, the 3rd. harmonic of the input shaft speed. This frequency is also known to result from a 3-times-per-revolution variation in the pinion tooth cutting process (which was later altered).

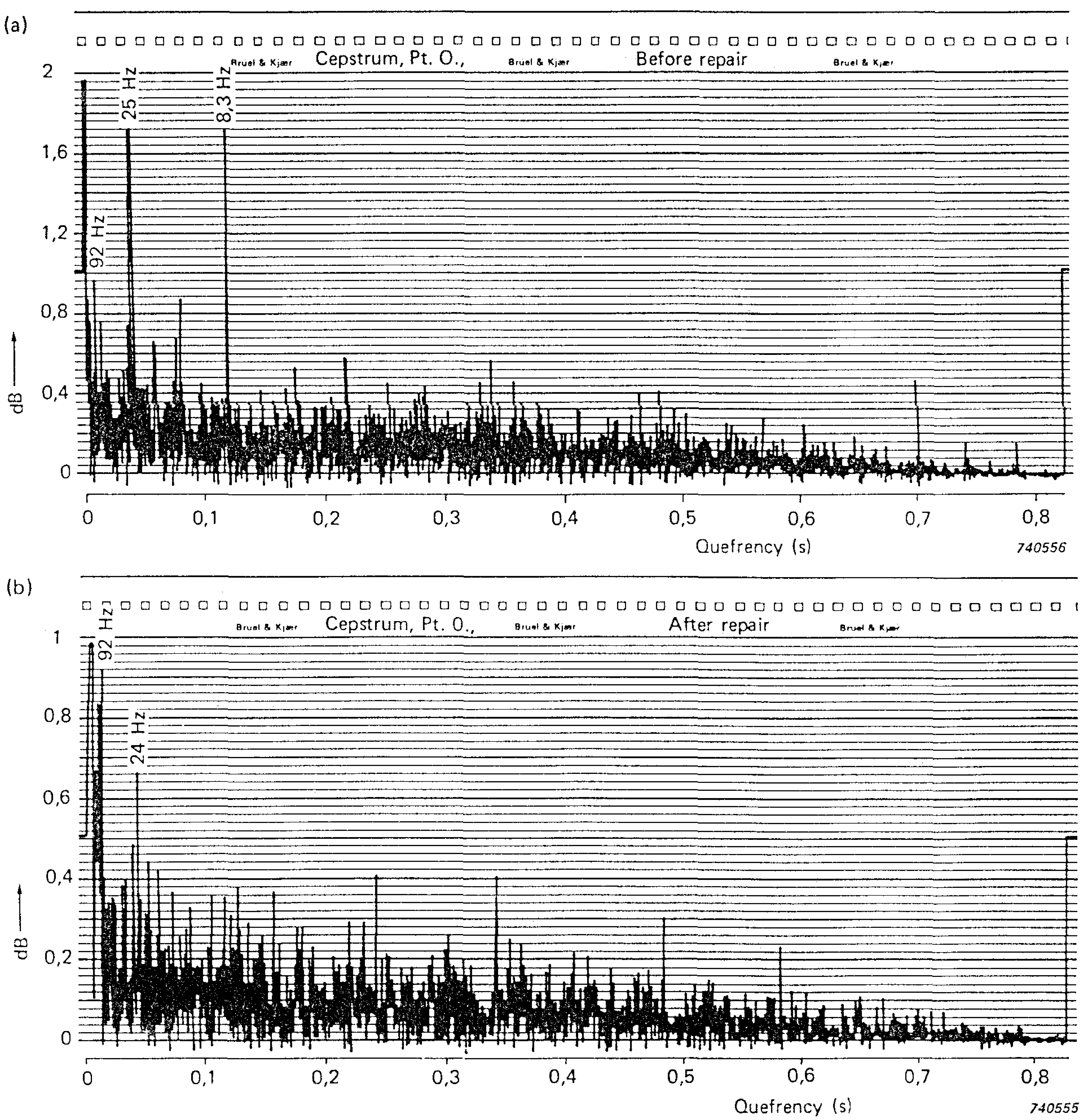


Fig. 14 Cepstra corresponding to Fig. 13
(a) Before Repair
(b) After Repair

This third harmonic component is approximately $2^1/2$ times larger in the cepstrum "before repair" (Fig. 14(a)) and is the dominant spacing evident in the spectrum (Fig. 13(a)), but the most remarkable increase is in the component corresponding to the pinion rotational speed itself (8,3 Hz) which has increased many times, and is as large as that at 25 Hz.

Conclusion

Only experience would tell at what point the modulation is serious, but the cepstrum technique appears to be a sensitive means for detecting changes in the spectrum not immediately obvious to the eye.

The effects of many factors still have to be investigated, such as noise level in the spectrum, filter bandwidth and shape, and sideband spacing, before anything can be said about absolute levels in the cepstrum.

However, it is thought that even at this stage, changes occurring in time in cepstra made under identical conditions should be significant, and at the very least can be used in conjunction with the spectrum to detect changes in the latter not immediately obvious to the eye.

It is thought that the major benefit will be more advance warning of impending failure thus giving more time for planning maintenance shutdowns, but it should also be a valuable diagnostic technique for detecting and curing sources of modulation at the machine development stage.

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APPENDIX A

FAST FOURIER TRANSFORM (FFT)

Virtually all Fourier analysis is based on the fourier integral pair

$$
\begin{aligned}
F(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\
f(t) &= \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega
\end{aligned}
\tag{1}
$$

which defines the complex frequency component $F(\omega)$ of the time function $f(t)$ for each angular frequency ω.

It is possible to write discrete equivalents of these equations as follows:

$$
\begin{aligned}
F(k) &= \frac{1}{N} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-jkn\,2\pi}{N}} \\
\text{and} \quad f(n) &= \sum_{k=0}^{N-1} F(k)\, e^{\frac{jkn\,2\pi}{N}}
\end{aligned}
\tag{2}
$$

which are known as the Discrete Fourier Transform (DFT). This transform pair has very similar properties to (1) with the exception that it is discrete and periodic in both time and frequency domains.

To calculate all N coefficients $F(k)$ directly according to (2) requires N^2 complex operations, but the FFT algorithm has been devised to obtain the same values in $N \log_2 N$ operations, a considerable saving.

One implementation of the algorithm, written in Fortran IV is the following program "NLOGN" obtained from Ref. 10.

```
      SUBROUTINE NLOGN(N,X,SIGN)
      DIMENSION M(25)
      DIMENSION X(2)
      COMPLEX X,WK,HOLD,Q
      LX=2**N
      DO 1 I=1,N
1     M(I)=2**(N-I)
      DO 4 L=1,N
      NBLOCK=2**(L-1)
      LBLOCK=LX/NBLOCK
      LBHALF=LBLOCK/2
      K=0
      DO 4 IBLOCK=1,NBLOCK
      FK=K
      FLX=LX
```

```
      V=SIGN*6.2831853*FK/FLX
      WK=CMPLX(COS(V),SIN(V))
      ISTART=LBLOCK*(IBLOCK-1)
      DO 2 I=1,LBHALF
      J=ISTART+I
      JH=J+LBHALF
      Q=X(JH)*WK
      X(JH)=X(J)-Q
      X(J)=X(J)+Q
2     CONTINUE
      DO 3 I=2,N
      II=I
      IF(K.LT.M(I)) GO TO 4
3     K=K-M(I)
4     K=K+M(II)
      K=0
      DO 7 J=1,LX
      IF(K.LT.J) GO TO 5
      HOLD=X(J)
      X(J)=X(K+1)
      X(K+1)=HOLD
5     DO 6 I=1,N
      II=I
      IF(K.LT.M(I)) GO TO 7
6     K=K-M(I)
7     K=K+M(II)
      IF (SIGN.GT.0.0) RETURN
      DO 8 I=1,LX
8     X(I)=X(I)/FLX
      RETURN
      END
```

The basic algorithm assumes that N is a power of 2 and that the input data are complex. An N-point complex data sequence f(n) gives N complex frequency components ranging from zero frequency (1st value) to just less than the sampling frequency for the original data points.

If the input data are real numbers, however, then half the storage space will be used for storing zeroes. Moreover, in that case the second half of the spectrum will be redundant since it is conjugate even about the Nyquist frequency (half the sampling frequency) and can thus be derived from the first half.

These inefficiencies are removed by using the following subroutine "HALF" based on an algorithm given in Ref. 11.

```
      SUBROUTINE HALF(NS,X,SN)
      DIMENSION X(2)
      COMPLEX X,FI,A1,A2,W
      FI=(0.0,0.5)
      NC=2**NS
      N2=NC/2
      W=CEXP(2.0*3.14159*FI/FLOAT(NC))
      CALL NLOGN(NS,X,SN)
      C1=0.5*(REAL(X(1))+AIMAG(X(1)))
      CN=0.5*(REAL(X(1))-AIMAG(X(1)))
```

```
      X(1)=CMPLX(C1,0.0)
      X(NC+1)=CMPLX(CN,0.0)
      X(N2+1)=0.5*CONJG(X(N2+1))
      DO 2 N=2,N2
      MINUSN=NC+2-N
      A1=0.5*(CONJG(X(MINUSN))+X(N))
      A2= FI*(CONJG(X(MINUSN))-X(N))
      X(N)=A1
      A2=A2*W**(-N+1)
      A1=0.5*(X(N)+A2)
      A2=0.5*(X(N)-A2)
      X(N)=A1
2     X(MINUSN)=CONJG(A2)
      RETURN
      END
```

This algorithm takes a sequence of N real data points and transforms them as N/2 complex points (using NLOGN), then manipulates the result to obtain the first half of the spectrum of the original data sequence. The output array contains N/2 + 1 values ranging from zero frequency (1st value) to the Nyquist frequency (last value). The zero frequency and Nyquist frequency components are necessarily real numbers (imaginary part zero), but the components in between are complex numbers of the form "a + jb" containing information of both amplitude and phase. The amplitude spectrum can then be determined as the length of each vector $\sqrt{a^2 + b^2}$ and the power spectrum as the square of this or $(a^2 + b^2)$.

The input data required for HALF, (NS, X and SN) have the following requirements:

NS – this indicates the size of the transform performed by NLOGN and is equal to $\mathrm{Log}_2\ (N/2)$ where N is the number of real data values in the input sequence.
eg. for a 1024 point transform, N = 1024 and NS = 9.

X – This is the input array of N/2 complex points and must be equivalenced in an "EQUIVALENCE" statement in the main program to the input array of N real data points. The result of the transform is also stored in X, and since this includes N/2 + 1 complex values the true dimension of X must be allowed for in the main program (the stated dimension of X in the subroutine is purely formal).

SN – This parameter must always be –1.0 and indicates in NLOGN that the transform is a forward transform.

Possible pitfalls in using the FFT algorithm are described in Ref. 13.